A Categorical Representation of Artificial Life

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ABSTRACT

While artificial intelligence (AI) is perceived as a major advance in machine research, less attention has been paid to artificial life, an attempt to create a life-form from computers and robots. There are difficulties in that there is no universal acceptance of what constitutes life or indeed a fundamental definition of biological organisms. There is certainly the need to employ a wider view of what constitutes life from a universal viewpoint than from the Earth alone. This requires a philosophical method that can provide universal constructions. ALife, as Artificial Life has become known, is split into three major approaches: Soft (software), Hard (hardware) and Wet (biochemistry). We concentrate on the Soft Approach, which relates strongly to the emerging subject of metabiology, examining the works of Rosen and Whitehead to provide a philosophical basis within a categorical framework. This metabiological framework is extended to handle AI by changing the basic constructions from Cartesian products to tensor products, using monoidal categories. Monad constructions are used for processing the monoidal categories, resulting in composition with higher-order 2-monads to achieve the complex processes required for artificial neural networks. The AI and metabiology are unified within a 3-pullback category over the nervous system.

1 Artificial Life

The basic ideas for ALife (Artificial Life) were developed by Christopher Langton in 1986 [1]. Three basic types of ALife are recognised: Soft (software), Hard (hardware) and Wet (biochemistry), all to simulate biological processes, some of which we know but also those that we do not know about yet. ALife is not just about life as we know it: ALife is archetypal, looking for a broad definition, covering life in any sphere. In more detail the differences in approaches are 1) Soft: using processes to simulate life, mainly through computer models; metabiology is a Soft approach. 2) Hard: using hardware (chips) to simulate life; there is a partial connection with AI and massively-parallel chips; this area also links to robotics. 3) Wet: using biochemical processes such as wetware and synthesising DNA.

There is an International Society for Artificial Life, which advances the vision for the field to be widely known and valued. This includes contributing to the good of human society and the biosphere and understanding life as it might be. Their documents present a somewhat defensive posture as the subject has a chequered history. The subject certainly includes metabiology, which handles the complexity of biological systems, in terms of information processing and mathematical reasoning. Also covered in a software approach is a study of evolutionary processes. Hardware approaches include the creation of an artificial body, with techniques such as an organ on a chip, brain-like chips and cellular automata (photonic computing). Wetware approaches, for designing an organic computer, include an artificial organic brain, an embryo, a biological virus and living neurons. Alife is a young interdisciplinary collection of research activities aimed at understanding the fundamental behaviour of life-like systems by synthesizing that behaviour in artificial systems. Artificial life research seeks to synthesize the characteristics of life by artificial means, particularly employing computer technology.

In comparison with AI (Artificial Intelligence), ALife creates life-like processes in artificial systems, such as software, hardware, or chemical systems. AI attempts to replicate human-like intelligence in machines. There is considerable overlap: ALife is sometimes thought to be bottom-up while AI is top-down. AI in practice today is concentrated on machine learning and pattern matching using large datasets and very fast hardware e.g. the GPU (chips) of Nvidia. There are examples of ALife/AI overlap, such as neural nets where both ALife and AI model brain structures and mechanisms. Similarly in genetic algorithms, ALife and AI both model birth, evolution and death.

An ALife organism continuously changes its own components and states at each moment through interaction with the external world, while maintaining its own individuality in a cyclical manner.

Such a property, known as autonomy, has been formulated using the mathematical concept of closure. Ryuzo Hirota [2] used categorical monoids to represent closure, in an approach which would benefit from more detail.

Natural is an important word, both philosophically and mathematically; for instance in category theory it provides unique identification of any object through the 'unique up to natural isomorphism' property. The Oxford English Dictionary currently excludes humans from its definition of nature, describing it as "the phenomena of the physical world collectively; esp. plants, animals, and other features and products of the earth itself, as opposed to humans and human creations." So Alife is bigger than nature. As an all-encompassing term Nature (Natural) is a cornerstone of philosophy. Whitehead in Modes of Thought (Lecture 7, "Nature Lifeless.", 173-201, in: [3]) derides much of contemporary and earlier work on Nature, particularly by Newton, claiming it was deficient both in analysis and abstraction: details are often poorly worked out and abstractions are inappropriate. Whitehead also disliked the positivism and dualism of Descartes [4]. We now look in detail at Metabiology, the aspects of ALife concerned with information processing and modelling.

2 Metabiology

Biology has always been the poor relation of physics when it comes to mathematical constructions. Metabiology appears to have no clear definition and no pedigree, unlike metaphysics which has a connection to Aristotle [5]. This is partly due to the perceived difficulty of describing the complex organisms and their transitions that underpin biology. But another important factor has been the lack of tools at the appropriate conceptual level.

In prior attempts at defining metabiology, Arturo Carsetti [6] discusses the concept in terms of: Husserls' phenomenology, sensory intuition and categorical intuition [7], Gödel's sentences [8], Lawvere's Cartesian closed categories [9], and Atlan's higher-order (self-organizing) cybernetics [10]. Chaitin, an early worker in metabiology, attempted to prove Darwinian theory through algorithms [11]. Turing developed wavelike patterns for biology that are the chemical basis for morphogenesis [12].

Other work on metabiology includes the two important sources of firstly Robert Rosen, who in his book *Life Itself* [13], developed the concept of relational biology, using categorical functors to describe living systems and secondly Alfred North Whitehead, who in his books *Process and Reality* [14] and *Modes of Thought* [3], developed novel concepts of prehension, concrescence, and eternal objects. We will deal with these works in depth later.

It is first necessary to define what we mean by the term Meta. This is interpreted variously as 'after', 'behind', 'beyond' [Wiki, metaphysics] or specifically 'data that provides information about other data' as metadata, and 'the study of mathematics itself using mathematical methods' as metamathematics. In organic chemistry meta is thought of as 'in the middle' or 'between' in concepts such as the benzene ring with its meta substitution.

Metaphysics is not a term coined by Aristotle but has been used by successors to describe a body of texts, to be studied after the ones dealing with nature *per se*. For instance in philosophy of physics, we can study aspects such as causes, principles, being, and existence in terms of more general and abstract principles, perhaps wisdom, after the physics has been considered. The proponents of metamathematics include Frege, Hilbert, and Kleene, often looking for a foundation to mathematics by generalising proofs to provide metatheories. Gödel's incompleteness theorems showed limitations in proofs on axiomatic systems. Some aspects of metamathematics have been superseded by mathematical logic to some extent. Metadata is well established in computing science for mapping specific data definitions to general principles of data structuring. It is essential for interconnection of systems, providing interoperability [15]. We can have multiple levels of abstraction (meta-meta is better-better!).

With metabiology we should attempt to build a theory on general principles, abstracting from the specific. It should be based on general processes, applied to abstract relations, with clear logical principles. With relevance to biology, an organism is any biological living system that functions as an individual life form. All organisms are composed of cells as developed in cell theory. The cells could be single or a composition of cells. Any individual animal, plant, bacterium, etc. has various parts or systems that function together as a whole to maintain life and its activities. Basically an organism is a form o life composed of mutually interdependent parts that maintain various vital processes.

Processes may be movement, reproduction, sensitivity, nutrition, excretion, respiration and growth; birth and death; evolution reflecting the adaptations of organisms to their changing environments, resulting in altered genes, novel traits, and new species. In anticipation, according to Rosen [13], a natural system's causal entailment model will be able to accurately predict future behaviours of the original, natural system.

For mathematical properties, the most obvious is identity, which is assured as organisms are individual. There is an internal structure, which may not be modelled if organisms are considered to be encapsulated, but DNA structures are applicable for explorations in this area. Process is considered more important than structure, leading to an essentially categorical basis with the arrow the primary modelling construct.

There appear to be two strong rival candidates for metabiology in the literature:

1) Robert Rosen's fundamental look at biological systems in terms of Life Itself (his book [13]), perceived as anticipatory systems [16]. This work uses formal categorical functors in an innovative way that was not appreciated by pure mathematicians at the time.

2) Alfred Whitehead's philosophy developed in Process & Reality [14] for organisms for *being* and *becoming*, the former for creativity, the latter through concrescence. It is shown elsewhere that Whitehead may be considered to be using informal category theory [17].

2.1 Formalization of Robert Rosen's Work

Robert Rosen (1934-1998) was a prominent theorist in the areas of biology and biophysics at Buffalo and Dalhousie. He was influenced by Rashevsky's work on mathematical biophysics [18], creating a biotopology. Rosen believed in complex systems and was against reductionist and reconstructive methods. He did not subscribe to Descartes' idea that animals are efficient machines [4]. The complexity that he developed refers to the causal impact of organization on the system as a whole.

Rosen's presentation of his modelling framework – (M, R) systems where M is metabolism and R is repair – is highly abstract and he gives few working examples of biological systems. Rosen's publications in his lifetime include in 1985, Anticipatory Systems [19], and in 1991, Life Itself [13]. Published posthumously were: in 2000, Essays on Life Itself [20], and in 2012, a second edition of Anticipatory Systems [21]. Rosen produced a modelling relation in 1991 as diagram 7F.1 in *Life Itself* as shown in Figure 1.

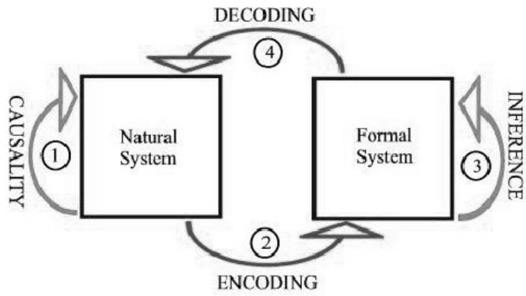

Figure 1. Modelling a Natural System with a Free System, from Rosen [13].

We have adapted Rosen's diagram slightly into Figure 2, replacing the arrow *Inference* by *Implication* and making it more obvious that the arrows *Implication* and *Causality* are endofunctors, that is functors where the type of the source and the target is equivalent.

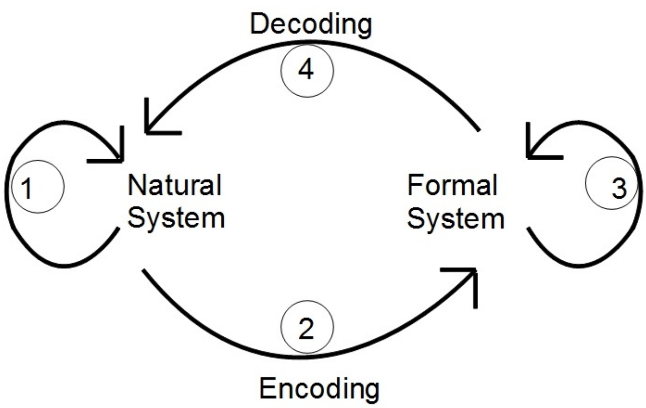

Figure 2. Modelling a Natural System with a Formal System, adapted from Rosen [13] Arrows: ① causality, ③ implication, ②, ④ as labelled

Rosen's categories are Natural System and Formal System and his functors in our view are:
Causality – cause and effect – ①;
Encoding – representation in model (free functor) – ②;
Implication – inference – ③;
Decoding – verification of model (underlying functor) – ④.

The arrow ① and the composition of arrows ④ ∘ ③ ∘ ② should be equivalent if the diagram can be said to be commutative. Rosen had difficulties in his earlier work in showing how the hybrid diagram of biological and mathematical equivalences commutes and indeed posed the question: does ① = ④ ∘ ③ ∘ ②? Such doubt reduced the acceptance of his claims to be using category theory, which would guarantee commutativity of the different paths. The property of adjointness follows from the axioms of category theory, which is discussed later. Rosen offered a formal translation using the block diagram approach of the general and logical theory of automata of McCulloch, Pitts, and von Neumann [22]. Kineman took Rosen's ideas and developed the idea of a Relational Holon. Bidirectional arrows are introduced between Encoding and Decoding and the Formal System is renamed Natural Model [23], as shown in Figure 3.

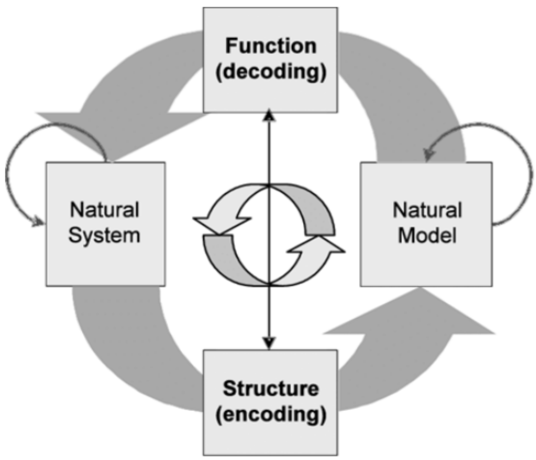

Figure 3. Relational Holon based on Rosen's modelling relation

Heather & Rossiter in work on anticipatory systems in 2009 [24] represented Rosen's work in Figure 4 by a two-way mapping of functors between categories $\mathfrak{NS}$ and $\mathfrak{FS}$ (in Gothic as large categories) as the adjointness ② ⊣ ④ <②, ④, η, ϵ > where ② is left adjoint to ④, ④ is right adjoint to ②, and the 4-tuple holds respectively functors ② and ④, η as unit of adjunction, and ϵ as counit of adjunction.

The unit and counit of adjunction for the objects in Figure 4 are defined in the triangles in Figure 5. These show that $\eta : NS \longrightarrow$ ④② NS and $\epsilon :$ ②④$FS \longrightarrow FS$.

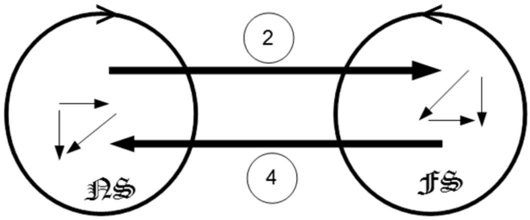

Figure 4. Two-way Mapping of Functors ② and ④ between categories 𝔑𝔖 and 𝔉𝔖

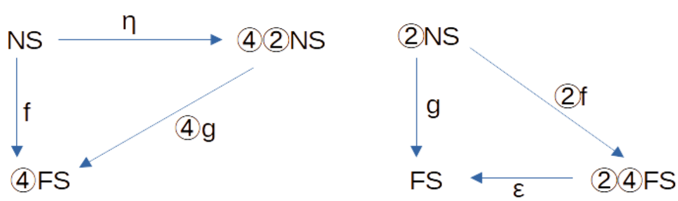

Figure 5. Unit and Counit of Adjunction for the Adjointness ② ⊣ ④

Rosen was pioneering in that he showed the potential for modelling fundamental biological processes in terms of mathematics. His work enables a comparison of biological causality with formal inferential modelling, which is anticipatory in nature. Ideally inferences in the model and the natural causality give equivalence but the introduction of adjointness into Rosen's framework by Heather & Rossiter widens the scope of the model to handle perturbations.

2.2 Formalization of Alfred N Whitehead's Work

We consider the other pioneer in modelling biological processes was Alfred North Whitehead but it is important to stress his work was more philosophical in nature and predated the establishment of category theory in 1945 [25]. In his book Process & Reality in 1928 [14] Whitehead develops the idea of feelings, based upon concrescence (physical realisation) of prehensions (capture, merger or ingression of objects) of eternal objects/actual entities.

In more detail feelings are sentient, not necessarily conscious. Prehension (being) does not necessarily lead to a concrete new entity, which is achieved through concrescence (becoming): when a new entity does result, we have emergence and potential evolution as per Darwin. Entities are described as real: they exist, as individuals; they are atomic; they are particular, so can be singled out or identified. Entities can be joined together as a nexus, giving a union of similar entities, termed an ordered society. Entities are also classified: every entity should be a specific instance of one category of existence ([14] at p.20). Entities can be seized by prehension (product or coproduct relatedness) to give a subjective form. Some entities are described as eternal objects: any entity whose conceptual recognition does not involve a necessary reference to any definite actual entities of the temporal world is called an eternal object. Eternal objects are therefore time-invariant or atemporal. Eternal objects are potentials; they could be an object like the number 2, but also include sensory qualities, like colours and tactile sensations; conceptual abstractions like shapes; numbers; moral qualities; physical fundamentals; feelings like an emotion, adversion, aversion, pleasure or pain; qualia such as artistic performances or a scientific paper. Eternal objects can participate in conceptual prehension in the form of feelings.

Looking at prehension in more logical terms, Whitehead says that every prehension consists of three factors: (a) the 'subject' which is prehending, namely, the actual entity in which that prehension is a concrete element; (b) the 'datum' which is prehended; (c) the 'subjective form' which is how that subject prehends that datum. This is a data relationship, either $\times$ or $+$, involving a pullback or pushout category respectively: $(c) = (a) \times_{(b)} D$ for the pullback, where D is a third entity introduced by us to complete the diagram; (c) is the subjective form, which can be defined in terms of a Cartesian closed $(\times)$ (or Cocartesian closed $(+)$) category. Figure 6 shows the Cartesian form.

In this diagram, the subjective form of Whitehead is the diagram $A \times_B D$ where A = (factor (a) above), B = (b), (c) = $A \times_B D$, D is the introduced entity. Prehension is a process by which an actual entity, or prehending subject, becomes itself by appropriating elements from other actual entities. The becoming of an actual entity occurs through a concrescence of prehensions. Satisfaction is a final phase of concrescence (or the process of integration of feeling), in which prehensions are integrated into a concrete unity.

in category theory concrescence is represented with further constraints as the locally Cartesian closed category $LCCC$ of Figure 7 with adjointness in the functors between the product $A \times_B D$ and B. In the hyperdoctrine of Lawvere [9], there is adjointness between

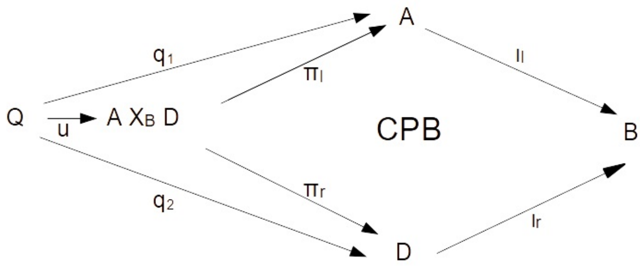

Figure 6. Category CPB: the Limit Diagram of a Category C: the Pullback $A \times_B D$

the existential, diagonal and universal functors: $\exists \dashv \Delta \dashv \forall$. Adjointness gives viability, enabling concrescence to occur. In Whitehead's terms prehension gives relatedness (being) and concrescence gives viability of relations (becoming). In categorical terms pullbacks give relations and adjointness gives viability.

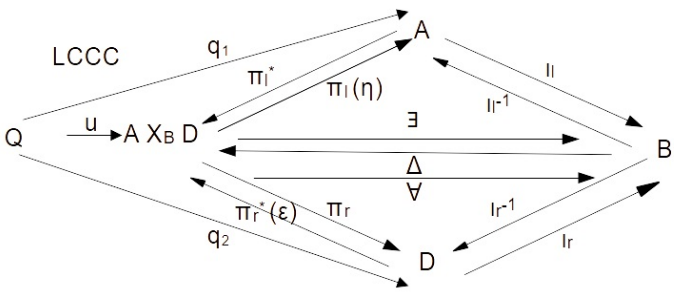

Figure 7. The locally Cartesian closed category LCCC: Adjointness in the functors between the product $A \times_B D$ and B

The nexus of Whitehead is a defining characteristic. A nexus enjoys 'social order' where (i) there is a common element of form illustrated in the definiteness of each of its included actual entities, and (ii) this common element of form arises in each member of the nexus by reason of the conditions imposed upon it by its prehensions of some other members of the nexus, and (iii) these prehensions impose that condition of reproduction by reason of their inclusion of positive feelings of that common form. Such a nexus is called a 'society' and the common form is the defining characteristic of the society. The notion of defining characteristic is allied to the Aristotelian notion of 'substantial form'. Thus the nexus forms a single line of inheritance of its defining characteristic.

In category theory a nexus is a sum, represented in general terms by a colimit or a pushout, a similar construction to a pullback but representing disjunction rather than conjunction. In the diagram the sum is of actual entities in a Society S with a line of inheritance INH in the context of the defining characteristic DC. The sum as a nexus or society is a pushout $S +_{DC} INH$, as shown in Figure 8, representing those entities with a common defining characteristic augmented with their position in the line of inheritance.

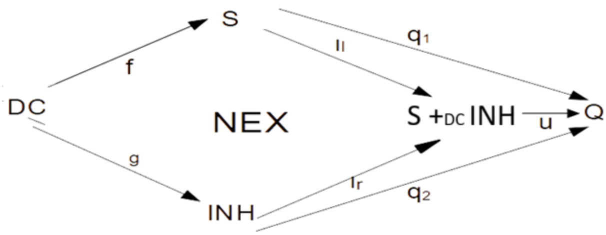

Figure 8. The Colimit Diagram of Category NEX: the Pushout $S +_{DC} INH$ as a social order on Society S with shared Defining Characteristic DC in a line of inheritance INH

One of Whitehead's most fruitful creations was the Category of the Ultimate([14], at p.21): "The ultimate metaphysical principle is the advance from disjunction to conjunction, creating a novel entity other than the entities given in disjunction. The novel entity is at once the togetherness of the 'many' which it finds, and also it is one among the disjunctive 'many' which it leaves; it is a novel entity, among the many entities which it synthesizes. The many become one, and are increased by one. In their natures, entities are disjunctively 'many' in process of passage into conjunctive unity. This Category of the Ultimate replaces Aristotle's category of 'primary substance'."

This category indicates a tension between × (conjunction) and + (disjunction) which featured strongly in our ANPA paper on music [26]. Aristotle's category of 'primary substance' is extensional while his secondary substance is intensional. Intension is an inherent part of each category through Dolittle diagrams as illustrated in this ANPA paper on music.

We consider the category of the ultimate to be a topos in mathematical category theory, represented by the category TOP where there is a tension between times (prehension) and plus (nexus) [17]. Here we show in Figure 9 the canonical form *TOPC* of this tension.

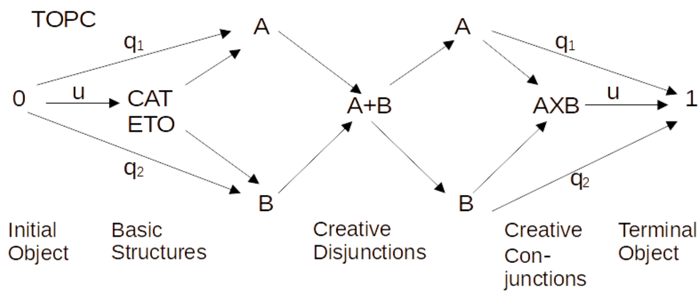

Figure 9. Canonical View *TOPC* of the Topos Category of The Ultimate *TOP*

The main results are that The Category of the Ultimate is a topos, handling the process *becoming*. In a single-substance philosophy, mind and matter are treated as one. The match between Whitehead's language and topos theory is close. The other special categories (Existence, Explanation, Obligations) expand and control the process of *becoming* and can be viewed as metaphysics or metabiology. Whitehead makes extensive use of some terms of his own, which we have related to category theory in Table 1:

Table 1. Whitehead's Terminology in Terms of Category Theory

Whitehead's term	Category theory
Prehension, relatedness	Cartesian/Cocartesian closed category
Subjective Forms	pullback, pushout
Nexus, togetherness	Cocartesian closed
Concrescence	adjointness (Locally Cartesian closed category)
Obligations	underlying functor (in adjointness)
Satisfaction	adjoint conditions realised
Category of the Ultimate	topos (Lawvere's hyperdoctrine)

Whitehead's potential is that emergence and evolution can be handled through prehension and concrescence respectively. For causality, prehensions can be nested, with a subjective form from one prehension becoming the object (datum) of another prehension, giving subject-object pairs. The canonical form of the topos diagram shown earlier in Figure 9 gives the potential for nested conjunctives and disjunctives. Rosen's approach is more model-oriented. From the biology viewpoint, both handle causality. Whitehead additionally handles evolution and Rosen additionally handles anticipation. From a formal viewpoint, both are amenable to category theory formalisation. From the meta viewpoint, Rosen has a simpler mechanism including model handling while Whitehead's principles are not always explicit. From the organism structure viewpoint, both are process orientated. The conclusion is that the variety gained by the two approaches should be valued.

3 Development by Whitehead of his Ideas

Whitehead expanded on his earlier ideas in Process & Reality, in Nature Alive, Chapter 8, pp 202-232, in Modes of Thought ([3], at pp 205-206). Whitehead establishes process as an essential concept, in particular prehension: "the notion of life implies a certain absoluteness of self-enjoyment. This must mean a certain immediate individuality, which is a complex process of appropriating into a unity of existence the many data presented as relevant by the physical processes of nature. Life implies the absolute, individual self-enjoyment arising out of this process of appropriation. I have, in my recent writings, used the word 'prehension' to express this process of appropriation. Also I have termed each individual act of immediate self-enjoyment an 'occasion of experience'. "

Whitehead identifies three characteristics of life: 1) Self-enjoyment, 2) Creative activity, 3) Aim. Self-enjoyment is prehension and Aim evidently involves the entertainment of the purely ideal so as to be the directive of the creative process. The enjoyment belongs to the process and is not a characteristic of any static result. "The aim is at the enjoyment belonging to the process" (p.208). Whitehead felt that the notion of creation, underpinned by aim, is essential to the understanding of nature. "By this term 'aim' is meant the exclusion of the boundless wealth of alternative potentiality, and the inclusion of that definite factor of novelty which constitutes the selected way of entertaining those data in that process of unification. The aim is at that complex of feeling which is the enjoyment of those data in that way. 'That way of enjoyment' is selected from the boundless wealth of alternatives. It has been aimed at for actualization in that process." (pp 207-208).

There are ten years between Whitehead's ideas in Process &

Reality in 1928 [14] and Modes of Thought in 1938 [3]. In 1928 Whitehead developed the idea of feelings, based upon concrescence of prehensions arising from ingression of eternal objects/actual entities. In 1938 there is no explicit concrescence. Feelings are developed into those prehensions that become actualizations. Feelings are the concrescence of prehensions but the concrescence is now more by choice than by viability. In both texts feelings are sentient, not necessarily conscious, and prehension (being) does not necessarily lead to a concrete new entity (becoming, via concrescence).

With respect to our earlier formalism, we need to modify the interpretation of Figures 6 and 7. The term D was originally introduced by us as a necessary feature to complete the pullback diagram, without any clear meaning from the Whitehead text of 1928 [14]. We can now apply the meaning 'Aim' to D from the 1938 text [3]. So semantically Figures 6 and 7 are now realised as the subjective form $A \times_B D$, the product of the subject prehending A and the aim D, in the context of the data being prehended B. The introduction of aim in the later text has thus assisted our formalism by giving a clear meaning to D. The whole diagram represents self-enjoyment.

Looking at other changes between the two texts, the Category of the Ultimate does not appear in the later Modes of Thought. Whitehead had evidently not realised its fundamental nature. However, Whitehead in Modes of Thought does supply an alternative more abstract idea: "[life] must mean a certain immediate individuality, which is a complex process of appropriating into a unity of existence the many data presented as relevant by the physical processes of nature." (p.205). The unity of existence is considered to be a topos TOP with identity 1_{TOP}. We have given above in Figure 9, a canonical view of the Category of the Ultimate as the topos category TOP showing in greater detail the tension between times (prehension) and plus (nexus). The topos does represent a unity of existence with, within itself i.e. intra-topos, the prehensions representing self-enjoyment through the conjunctions representing prehension.

Creativity is moving from one topos to another, through an inter-topos process such as evolution or emergence: $F : 1_{TOP} \longrightarrow 1_{TOP'}$ where F is a free functor. On a lesser scale the free functor could modify the extension of the creative disjunctions and conjunctions within a topos. We consider intra-topos processes to include self-enjoyment, adaptability and survival. Satisfying the aim requires restricted creativity, qualified again by aim, which is achieved by adjointness with the adjoint pair: $F : 1_{TOP} \longrightarrow 1_{TOP'}$, $G : 1_{TOP'} \longrightarrow 1_{TOP}$. G is the underlying functor, enforcing the qualification. The whole is written as $F \dashv G$ and the 4-tuple $< F, G, \eta, \epsilon >$ where η is the unit of adjunction and ϵ is the counit of adjunction. The overall process is a monad, in this case an endofunctor, such as T where $T = GF$ for restricted creativity, as shown in Figure 10. As a general qualification the structures we have defined are Cartesian, assuming simple data structures without complex internal structure.

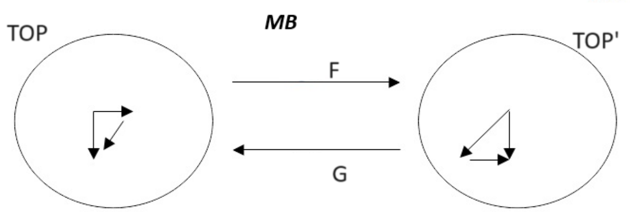

Figure 10. Restricted Creativity: ***MB***: The Monad T for the adjoint $F \dashv G$

To conclude this section, Whitehead's Process & Reality provides a philosophical basis for category theory. This is an alternative to the classical pure mathematical/set-based basis. Process & Reality gives a much richer realism than set theory, facilitating process as a single substance approach and providing an enriched type system, including complex metadata in the three additional special categories, a move towards ALife. Whitehead's Modes of Thought provides a more abstract view of nature than that given in Process & Reality but it is consistent with his earlier ideas and the introduction of Aim facilitates our formalisation. The adjoint structure for restricted creativity leads to a monad as in the music composition in our earlier work. We next look at AI, another vital component of ALife.

4 Artificial Intelligence

We consider neural nets as the main practical technique at present for AI. In Artificial Neural Nets (ANN), the fundamental object is the artificial neuron, a receptor or a transmitter of signals, often an algorithm. A signal between neurons, termed a synapses, can be weighted. Neurons are assembled in layers with the opening layer receiving inputs and the closing layer providing an output. The middle layers are hidden. A neuron can have many inputs but usually just one output, which can branch into multiple neurons.

Theories for ANNs are inspired by biology. Nodes with connections, which have weights, are a popular choice. The weights are adjusted in training with a universal function approximator, which approximates any continuous function to an arbitrary degree of accuracy. Such a system must learn and represent a vast range of functions. So-called deep learning involves multiple hidden layers. The output of each neuron is computed by some non-linear function of the totality of its inputs. The output is a real number. The AI existing at the present time such as in ANNs is classified technically as Weak AI: the ability to handle specific tasks for which intensive training has been provided. Strong AI is the ability to provide general intelligence at the level of a human; this capability does not exist at present and there is as yet no accepted formalism to handle it. Super AI is the ability to provide intelligence above the human level and is even further away. The requirement for a theory needs to be emphasised. A theory aids teaching and understanding and adds confidence in results through proofs. A theory can also emphasise commonality with other approaches, aid extensions, emphasise assumptions, and ease implementation. But theory has to be appropriate for the target.

4.1 Formalisation of AI with Category Theory

A number of formalisms have been suggested, including machine learning, adaptive learning, and statistical techniques such as probability, statistics, Bayesian with weights, linear and non-linear algebra, and dynamical systems theory. Category theory appears to be particularly appropriate for ANNs as it provides inherent facilities for composition of layers, levels and two-way mapping.

There should be a connection between AI and category theory as AI (Artificial Intelligence) is mainly based on neural nets with the connections being dual multi-level directed graphs, based on models, ideally suited for the dual multi-level arrow constructions of category theory. However it should be said that multi-level ANNs (Artificial Neural Net models) are complex, a challenge to any theory.

Previous work on transactional systems such as databases, information systems and music, had identified the topos as the data structure of choice with Cartesian products and the monad as the computational unit, rolling the process forwards and backwards [27, 26]. Neural net systems cannot be handled by this approach as the data structures involve tensor products of more complex data structures such as vectors. Such data structures are represented by monoidal categories, best treated as 2-categories to handle the extra level in the data structure. A monad is still appropriate for handling the processing between the layers of the neural net but will now be in the form of a 2-monad to accommodate the extra level. The whole is now a 3-category with (arrows between arrows) between (arrows between arrows) as the activity in a neural net. It should be possible to combine our earlier approach using the topos with the current neural net work by introducing the concept of a monoidal topos where the Cartesian product is augmented by an additional tensor product.

So the conversion from Cartesian products to tensor products is an essential step in handling ANNs. Cartesian products relate whole entities to one another and alone do not provide the ability to process the fine structure of neural nets: we require tensor products to enable vector spaces to be subject to cross-product operations so that every element and subparticle is combined with every other element and subparticle. A monad is a monoid in the category of endofunctors, which is a functor from a category to itself. Often the endofunctor is actually a pair of adjoint functors $G \dashv F$ where $F : C \longrightarrow D; G : D \longrightarrow C; G \circ F : C \longrightarrow D \longrightarrow C$ as in much of our earlier work on information systems and music. A monad is a monoid in a particular setting and within category theory is an important level of abstraction. A monoid is defined algebraically as a set with an associative binary operation and an identity element, sometimes termed a neutral element, which leaves an object unchanged in the operation. Examples are multiplication $M \times M \longrightarrow M$ and addition $M + M \longrightarrow M$. In category theory a monoid comprises an object, a hom-set (identity arrows as in say a set), a composition morphism $(x, +, ...)$, and a unit (neutral) element. Examples are $(C, +, 0)$ for addition and $(C, \times, 1)$ for multiplication. Monoid categories can be enriched if the objects have complex internal structure or complexity beyond simple operations, such as with tensor product operations, for example $(C, \otimes, I)$, where I is the tensor unit. If we consider $A \otimes B$ where A and B are vectors, the operation performs pairwise multiplication of each component of each vector while the Cartesian product would pair each whole vector with another whole vector. Tensor products are usually formalised in a monoidal category, which can also handle Cartesian products. Tensor products give n-tuples as they can have many arguments. A monoidal category is a category C equipped with:

1. a functor $\otimes : C \times C \longrightarrow C$, out of the product category with itself, called the tensor product,

2. an object $1 \in C$ called the unit object or tensor unit,

3. a natural isomorphism $a : ((-) \otimes (-)) \otimes (-) \xrightarrow{\sim} (-) \otimes ((-) \otimes (-))$ with components of the form $ax, y, z : (x \otimes y) \otimes z \longrightarrow x \otimes (y \otimes z)$ called the associator,

4. a natural isomorphism $\lambda : (1 \otimes (-)) \xrightarrow{\sim} (-)$ with components of the form $\lambda x : 1 \otimes x \longrightarrow x$ called the left unitor,

5. a natural isomorphism $\rho : ((-) \otimes 1) \xrightarrow{\sim} (-)$ with components of the form $\rho x : x \otimes 1 \longrightarrow x$ called the right unitor,

such that the following two kinds of diagrams commute for all objects: the triangle identity as in Figure 11 and the pentagon identity as in Figure 12.

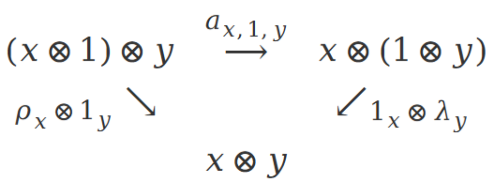

Figure 11. Triangle Identity for a Monoidal Category

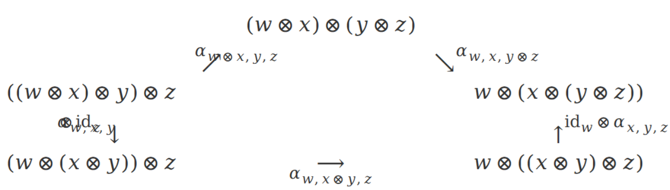

Figure 12. Pentagon Identity for a Monoidal Category

Our motivation here is to achieve non-linearity, not possible with basic categories, and to handle more complex mathematical structures. Monoidal categories are fundamental building blocks for higher categories and the modelling of complexity.

Monoidal categories involve an extra layer of arrows for the complex, internal structure so the arrow structure involves more levels. We need to move to a category of category structure. This brings in higher-order category theory, initially at the 2-category or double category level. n-cells are the basis of n-categories. We have met the concept before as an adjunction. It is also now known as a lens category. An n-category may contain a heterogeneous collection of basic cells: a 0-cell is objects (identity arrows), a 1-cell is arrows between objects (category), a 2-cell is arrows between arrows (natural transformation, adjunction), a 3-cell is (arrows between arrows) between (arrows between arrows) or adjunction arrows mapped onto adjunction arrows (termed a modification). In the conclusion of a PhD dissertation at Northumbria University, Sisiaridis used 3-cells for security in information systems ([28], Figure 6.24) as in Figure 13, where a modification (3-cell) $\mu : \sigma \Rightarrow \tau$ may modify the relation between two events. For example, for an event a that belongs or not to a consistent run R, it may be determined by examining μ if it is parallel with another event b.

The 4-cell and beyond have been used in knot theory. In an n-category there is no upper limit to n, which is the highest ranking cell in the category, if we have heterogeneous structures of cells. A category with just 0-cells (objects) is a set, a discrete category. A category with just 1-cells (arrows between objects) is an ordinary category. A category holding up to 2-cells (arrows between arrows) is a 2-category. A category holding up to 3-cells – (arrows between arrows) between (arrows between arrows) is a 3-category. These cells are shown in Figure 14.

In a 2-category (and above) unique up to natural isomorphism (composition) is no longer in general enforced except in a strong 2-category where composition and associativity are strictly applied throughout; such situations do not occur often in nature. In a weak 2-category or multicategory, associativity is restricted up to a particular level of isomorphism (1-isomorphisms for instance). In a multicategory the objects of an arrow are not single objects but a finite sequence of them. So an object can be a vector space. Respecting this structure requires a tensor product for multiplication. An operad is an operation taking a multicategory as input and producing a single output. So it can convert a vector space to

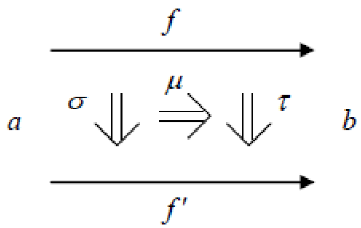

Figure 13. A modification (3-cell) μ that determines if two events are parallel or not

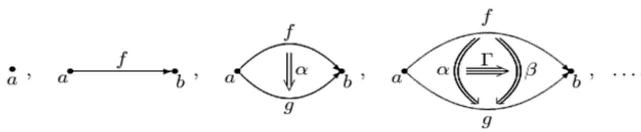

Figure 14. n-cells: $n = 0, 1, 2, 3$ from left to right respectively

a single output or a tensor product of a vector space to a single output. A lax monoidal category or skew monoidal category is a monoidal category in which the associator- and unitor- transformations are not required to be invertible, i.e. are not required to be natural isomorphisms (as they are for strict monoidal categories). This allows more flexibility in the mappings beyond isomorphisms. There are still some constraints. Typically there is a bias in one direction (left or right). A left-biased lax monoidal category has a binary tensor product $A \bigotimes B$, a unit I and constraints, but no inverse for the associator:

$$a \otimes (b \otimes c) \longrightarrow (a \otimes b) \otimes c; I \otimes a \longrightarrow a; a \otimes I \longrightarrow a$$

Relevant to the linear-non-linear adjunction is the superposition principle, also known as superposition property, which states that, for all linear systems, the net response caused by two or more stimuli is the sum of the responses that would have been caused by each stimulus individually. Category theory does have a non-linear construction: for example, a composition of adjoints, one tensor, one Cartesian, is termed a linear-non-linear (LNL) adjunction, as is any well-formed category including a topos. LNL is a property of intuitionistic logic where there is no closed world assumption [29, 30]. In a 2-Category, a linear-non-linear adjunction is for example an adjunction $L \dashv M$ between a Cartesian monoidal 2-category $\boldsymbol{M}$, resulting from $(M, X, \top)$, and a non-Cartesian symmetric monoidal 2-category $\boldsymbol{L}$, resulting from $(L, \otimes, 1)$ ([31], Chapter 7). The comonad $! = LM$ on $\boldsymbol{L}$ is the basis for categorical semantics of linear logic. The similarity between this diagram and that adapted from Rosen's in Figure 2 should be noted. Notation-wise we use bold italics for monad categories.

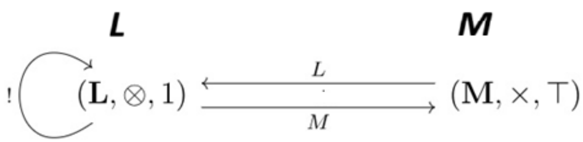

Figure 15. For 2-Categories $\boldsymbol{L}$, $\boldsymbol{M}$: Linear-non-linear Adjunction: $L \dashv M$

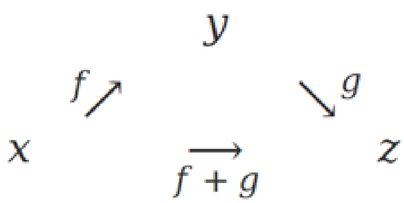

Figure 16. Triangle Inequality: $w(ID_X) = 0$ and $w(g \circ f) \leq (w)g + w(f)$

Weighted arrows increase the complexity of category theory. Lawvere [32] was the first to investigate this. He utilised a real number to represent the distance between two points, with curvature permitted, in a topology context so non-Euclidean, in what is now termed a Lawvere metric space. This space is defined to be a set X equipped with a distance function $d : X \times X \longrightarrow [0, \infty]$, sometimes portrayed as a triangle inequality, but now viewed more often as an example of an enriched category. The triangle inequality, in its most basic form, states that for any triangle, the sum of the lengths of any two sides must be greater than or equal to the length of the third side. The distance $d(x, z)$ between any two points x, z is no larger than the sum of the distances $d(x, y)$ and

$d(y,z)$ via any third point y. A weighted category has a number, for example a cost or length, and composition comes with a triangle inequality, as shown in Figure 16. So for such a category, to each morphism $f : A \longrightarrow B$ is assigned a non-negative real number (up to ∞) – the weight w(f) – such that $w(ID_X) = 0$ and $w(g \circ f) \leq (w)g + w(f)$ (the triangle inequality). For enriched weighted categories we define a weighted set to be a set X equipped with a function: $w : X \longrightarrow [0, \infty]$. Take a (short) function $f : (X, w_X) \longrightarrow (Y, w_Y)$ with rule $w_Y(fX) \leq (w_X)X$. A closed monoidal category is constructed by taking the tensor product of weighted sets in the Cartesian product of the sets (x,y) : $\forall (x,y) \epsilon (X,Y) : w(x \otimes y)((x,y)) = w_X(x) + w_Y(y)$. A symmetric closed monoidal category has LNL.

In earlier work, for the relation between a topos and a closed monoidal category, we used as our data structure, the topos or at least a Cartesian closed category. A closed monoidal category is equivalent to a Cartesian closed category when the object is a discrete set with no internal structure as the product is Cartesian rather than tensor. They are not equivalent if say the object is a vector space as the monoidal category involves a tensor product.

Cartesian closed categories and the topos are based on limits and colimits for universal constructions. We can also construct limits for monoidal categories, which respect the monoidal structure of tensor product and identity. The limit functor should be at least a lax monoidal functor, meaning it preserves the monoidal structure up to specified isomorphisms. In a sense our work with monoidal categories has built towards adapting the topos, as classically perceived for Cartesian products, to a monoidal topos, where the product in the topos is tensor rather than Cartesian. So we could write our topos more strictly as $(TOP, \otimes)$ rather than $(TOP, \times)$. Some monoidal topos are doubly-closed, representing a Cartesian part with non-linear intuitionistic logic and a tensor part with linear logic, realising a Linear-non-Linear construction. The topos that we use later under the control of a monad is a monoidal topos.

4.2 Applications of Monoidal Categories

There are many applications or perhaps to be more accurate, intended applications, of monoidal categories, including neural nets. Using monoidal coalgebraic metrics, Filippo Bonchi's team at the University of Pisa intend to establish a robust mathematical framework that extends beyond the metrics expressible in quantitative algebraic theories and coalgebras over metric spaces by shifting from Cartesian to a monoidal setting [33], as in this paper. Bonchi has since investigated the theory of traces for systems with non-determinism, probability, and termination, using monads, an abstraction of monoids [34]. Most of their results are based on the exciting interplay between monads and their presentations via algebraic theories. Spivak, a long-standing researcher on category theory applications, was involved in a team defining a monoidal functor from a category of parameterised functions to a category of update rules, using back propagation as in neural nets [35].

The Topos Institute is a mathematics research institute in Oxford with a focus on category theory. On their blogs they have 3 entries on monoidal categories: Evan Patterson, Unbiased monoidal categories are pseudo-elements; David Spivak, Promonoidal categories and wiring diagrams; Brandon Shapiro, A dynamic monoidal category for strategic games [36]. A common thread in these articles is their use of monoidal categories for representing higher-order categories.

In applications of monoidal categories to physics, Matthew Varughese investigated categorical quantum computing [37]. Under his supervisor Jamie Vicary of Oxford University, he researched the link between classical structures in *Rel* the category of finite relations and single valued classical structures in $FdVectF2$. He developed monoidal structures in $FdVectFp$ (dimension n) that are isomorphic to the finite field Fpn. The main result was that such monoidal structures always exist and each admits a special Frobenius algebra. This work also provides a background to the use of category theory in quantum computing, including symmetric monoidal categories, used in our work. A categorical approach to knot theory has also been pursued. Khovanov homology is a categorification of the Jones polynomial [38]. Khovanov constructs a bigraded cohomology theory of links whose Euler characteristic is the Jones polynomial, by developing a monoidal category with tensor product of morphisms defined by taking the disjoint union of surfaces. The approach is categorification, involving a literal translation of knot properties into categorical ones.

4.3 The Monad Abstraction

In relation to our earlier work in database transactions [27] and music [26], we used a monad to represent transactional process on a Cartesian closed structure with certain nice properties such as limits and a subobject classifier: a topos. The monad represented activity transactionally through an adjointness between the current state LYR and the next state LYR' with the topos representing the data structure. If we perform one 'Cycle' as shown in Figure 17 for adjointness IH $(H \dashv I)$, we can assess the unit of adjunction in LYR and the counit of adjunction in LYR' to ensure overall consistency as $\eta : 1_{LYR} \longrightarrow IH(LYR); \epsilon : HI(LYR') \longrightarrow 1_{LYR'}$. We are using functors H and I as free and underlying functors respectively and V as the monad $I \circ H$ to distinguish the expressions from those developed for metabiology. We can view a neural net as a transactional monad where each layer is a closed monoidal category with tensor products and weights, within a 2-category, with lax symmetry, relaxed isomorphism and associativity requirements giving non-linearity. Such a layer could be viewed as a doubly-closed monoidal topos, in keeping with our earlier work preferring the topos as a data structure. Between each layer is a goal-directed free functor and an underlying functor, enforcing back propagation which is an essential part of the training. Such a construction is a 2-monad, providing some transactional conditions are satisfied. Such monads can be composed if some housekeeping conditions are satisfied, particularly distributive ones such as the Kleisli-lift in Cartesian monads [39]. 2-monads have also been found to be useful in general architectures for categorical deep-learning, as applied to a broad range of neural net systems, including Recurrent Neural Nets (RNN), which handle serial data such as text [40]. This work is related to that of Symbolica, a company created to handle machine learning with category theory [41].

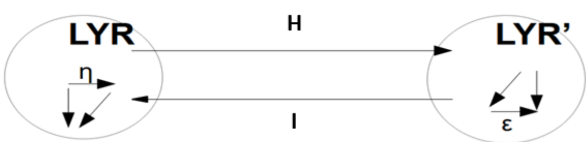

Figure 17. One 'Cycle' for adjointness $H \dashv I$ with $H : LYR \longrightarrow LYR'; I : LYR' \longrightarrow LYR$

We can compose 2-Monads as shown in Figure 18 to emulate an Artificial Neural Net $\boldsymbol{ANN}$ of the form $V''' \circ V'' \circ V' \circ V$ where $V = I \circ H$. Each layer LYR is a 2-category: arrows between arrows. Each 2-monad resides in a 3-category: (arrows between ar-

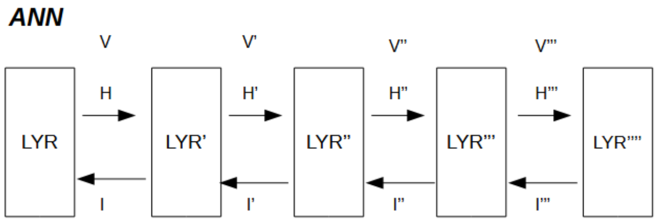

Figure 18. Composition of 2-Monads: $\boldsymbol{ANN} : V''' \circ V'' \circ V' \circ V$ where $V = I \circ H$

rows) between (arrows between arrows): 2-monads exist within a 3-category, which may be strict, fully weak, lax, pseudo, or co-lax. Pseudo preserves the structure up to a specified isomorphism, giving much scope for flexibility in strictness of associativity and isomorphism rules.

To conclude this section, category theory has developed from its simple Cartesian basis with natural transformations and adjointness. Monoidal categories provide tensor products, n-categories provide controlled relaxation of rules and 2-monads provide composition of adjoints between categories, within a 3-category. Such 3-categories begin to match the complexity of neural net systems. This is Weak AI. In future work we will look to increase the number of levels to handle Strong AI.

5 Artificial Life in Categorical Terms

We have analysed above two major components of artificial life: artificial intelligence and metabiology. Using Whitehead and Rosen as mentors, we developed a categorical framework for metabiology based on their philosophy. This gave essentially a Cartesian world with a topos TOP representing the tension between disjunction and conjunction in the natural world for aspects such as evolution, birth, life and death. In artificial intelligence, we developed the closed monoidal category or monoidal topos $\boldsymbol{ANN}$ to handle the tensor products, layers, weights, linear-non-linear logic, and other features of Artificial Neural Nets. Metabiology required 1-monads (or simple monads) and artificial intelligence required 2-monads, with more complex structures. To combine the two, we need to find some way to integrate the two processes.

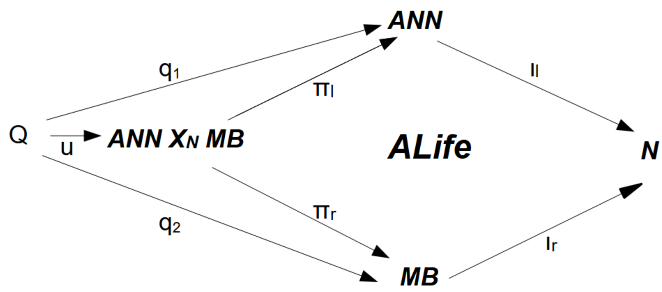

Figure 19. Category ***ALife***: the Limit Diagram for a 3-Category: the 3-Pullback $\boldsymbol{ANN} \times_N \boldsymbol{MB}$

The 2-monads are the AI, with complex categorical structure. The 1-monads are the metabiology, representing physical aspects of life. Both the 1-monads and the 2-monads can reside within a 3-category. It seems plausible to suggest that the intelligence component drives the metabiology, rather than the other way round. But there is obviously a two-way mapping in that the intelligence requires a physical link for actions and that the actions feed into how the intelligence operates. Monads are arrows, basically endofunctors, so we can construct relationships between them. An adjunction is possible but the difference in the types, including concerns for the complex dimensionality, makes this difficult to construct. A better construction, with this heterogeneity, is to consider a colimit or limit for the 3-category; such a limit as the 3-pullback is shown in Figure 19. ***N*** is the nervous system, a 1-monad, acting as the go-between the brain and the physical system. The AI is the 2-monads-category ***ANN*** and the metabiology is the 1-monad category ***MB***. It is tempting to make this a locally Cartesian closed category as then we can attain adjointness through the Lawvere hyperdoctrine [9] as in Figure 20. The arrow Δ, the diagonal, selects pairs of ***ANN*** and ***MB***, matching a nervous system property, from the products that occur in reality; the arrow $\exists$, the existential quantifier, maps from the product to a single nervous system property; the arrow $\forall$, the universal quantifier, maps from the product to a collection of nervous system properties. If there is adjointness $\exists \dashv \Delta \dashv \forall$ then we have consistency within the relationships, with unit and counit of adjunction in commuting diagrams, examples of which are in Figure 5 above. However, this diagram requires further work, in the direction of the mathematical research on locally Cartesian closed $(\infty, 1)$-categories which cater for achieving this type of construction in n-categories [42]. So for the present we return to Figure 19 as our optimum strategy. More information on pullbacks in higher-order categories can be found at NLAB [43].

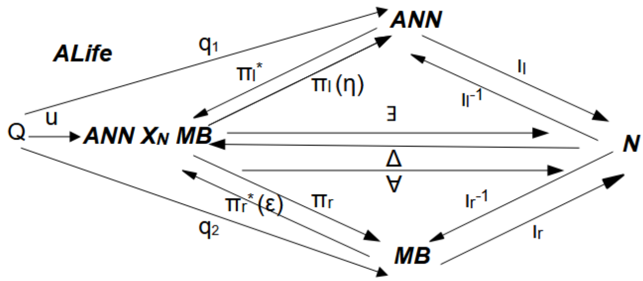

Figure 20. Putative Locally Cartesian Closed category: The Pullback Limit for the 3-Category ***ALife***: Adjointness $\exists \dashv \Delta \dashv \forall$ in the functors between the product $\boldsymbol{ANN} \times_N \boldsymbol{MB}$ and ***N***

6 Closing Remarks

There are still many problems to solve in AI. Weak AI, the practice today, can be achieved by sheer force of computing power, using many of the neural net techniques developed in the 1980s. A number of these techniques are black-box in the sense that internal parameters are optimised, in intensive training, without any overall formalism. It is very doubtful that Strong AI, where computers take on human characteristics, can be achieved without a greater understanding of how the human brain works and with a different mathematics that can formalise consistently over multiple levels. Category theory does have much potential here. A future paper will look at categorical models for Strong AI.

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