The Monad in Process-Relational Systems

Nick Rossiter
Visiting Fellow
Computing Science and Digital Technologies
Northumbria University

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Process-Relational Philosophy 1

- Whitehead's Process and Reality introduces many of the concepts of metaphysics.
- Later workers, including Robert Mesle, Margaret Stout and Mary Follett, have used the ideas of Whitehead to formulate the process-relational philosophy.
- Such a philosophy has been applied in a social context to handle creativity, Becoming, imagination and experience.
- In a language context, the same philosophy has been applied to ontology or Being.

Process-Relational Philosophy 2

- The process-relational philosophy considers that the world can be thought of a collection of interrelated processes,
  - rejecting the Cartesian dualism of Descartes, and
  - favouring the dynamic process (flux) of Heraclitus.
- Such a philosophy satisfies current requirements in computer science and information systems but has often been difficult to achieve.

Problems in Computing Science

- The basis of much of computer science is set theory,
  - provides adequately the static (Being)
  - but is restricted to process as function.
- Further, handling the logical types across the static and process components in an integrated manner is very difficult in practice, a problem encountered by Russell and Whitehead in their series on set theory, Principia Mathematica.
- A single-level approach is inadequate for the complexities of information systems.

Process and Reality

- Much of Whitehead's Process and Reality can be considered as informal category theory
  - preceding the later developments in pure mathematics, starting in the 1940s by such workers as Eilenberg and Mac Lane (EML Category Theory)
- For instance Whitehead's category of prehension, or grasping, corresponds to the categorial adjunction.
- Other examples are that Whitehead's category of the ultimate corresponds to the topos and his category of existence to the Cartesian Closed category.

Process-relational and Category Theory

- In this paper we consider how the process-relational philosophy, naturally arising from Process and Reality, can be considered formally in category theory by the monad, which relates inputs and outputs through adjointness.
- The monad operates on a category, such as a topos, over three-levels, providing the necessary closure of being defined as unique up to natural isomorphism.
- The term monad is very 'old' but was made better known by Leibniz. We have made a comparison of the various usages of the term, including its use today in mathematics and computer science.

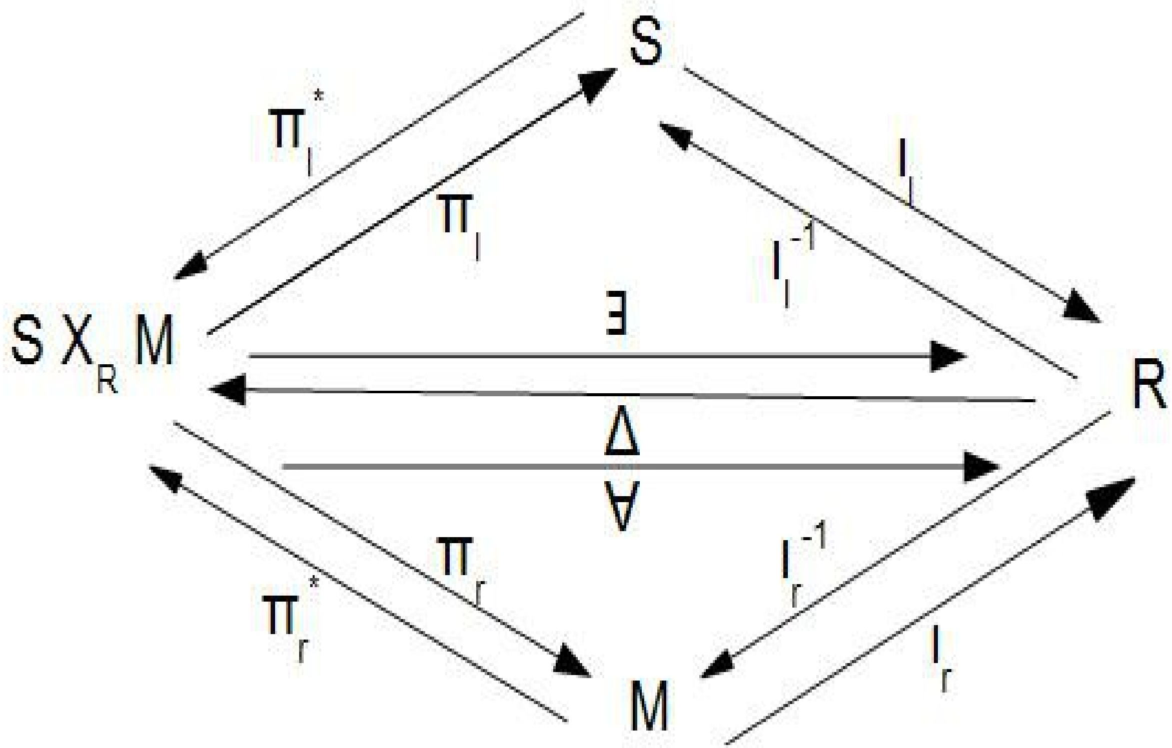
The Topos – Structural Data-type

- Is a Cartesian Closed Category (CCC)
  - Products; Closure at top; Connectivity (exponentials); Internal Logic of λ calculus; Identity; Interchangeability of levels
- If we add:
  - Subobject classifier
  - Internal logic of Heyting (intuitionistic)
  - Reflective subtopos (query closure)
- We get a Topos

Structural Examples

- Student Marks
  - Simple (single pullback)
- Bank Transactions
  - Simple (single pullback)
  - Simple pasted (2 pasted squares, 3 pullbacks)
  - Complex (5 pasted squares, 10 pullbacks)
  - Complex structure (5 pasted squares, not valid pullback)

Pullback - Single Relationship
Student Marks



Pullback - Single Relationship Constraints

- SX_R M (Student X_{Result} Mark)
- Logic of adjointness: $\exists \dashv \Delta \dashv \forall$
  - Δ selects pairs of S and M in a relationship in context of R
  - Such that $\exists \dashv \Delta$ and $\Delta \dashv \forall$
  - Termed by Lawvere as a hyperdoctrine
- Projections π are from product, left and right (dual π*)
- Inclusions ι are into sum S+M+R, left and right (dual ι^{-1})
- S, M, R are categories, with internal pullback structure, giving recursive pullbacks

Recursive Pullbacks

A node of a
pullback may
itself
be a pullback

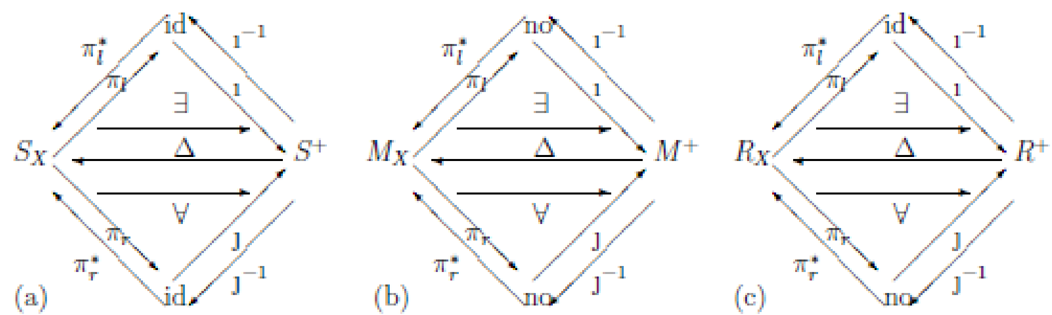
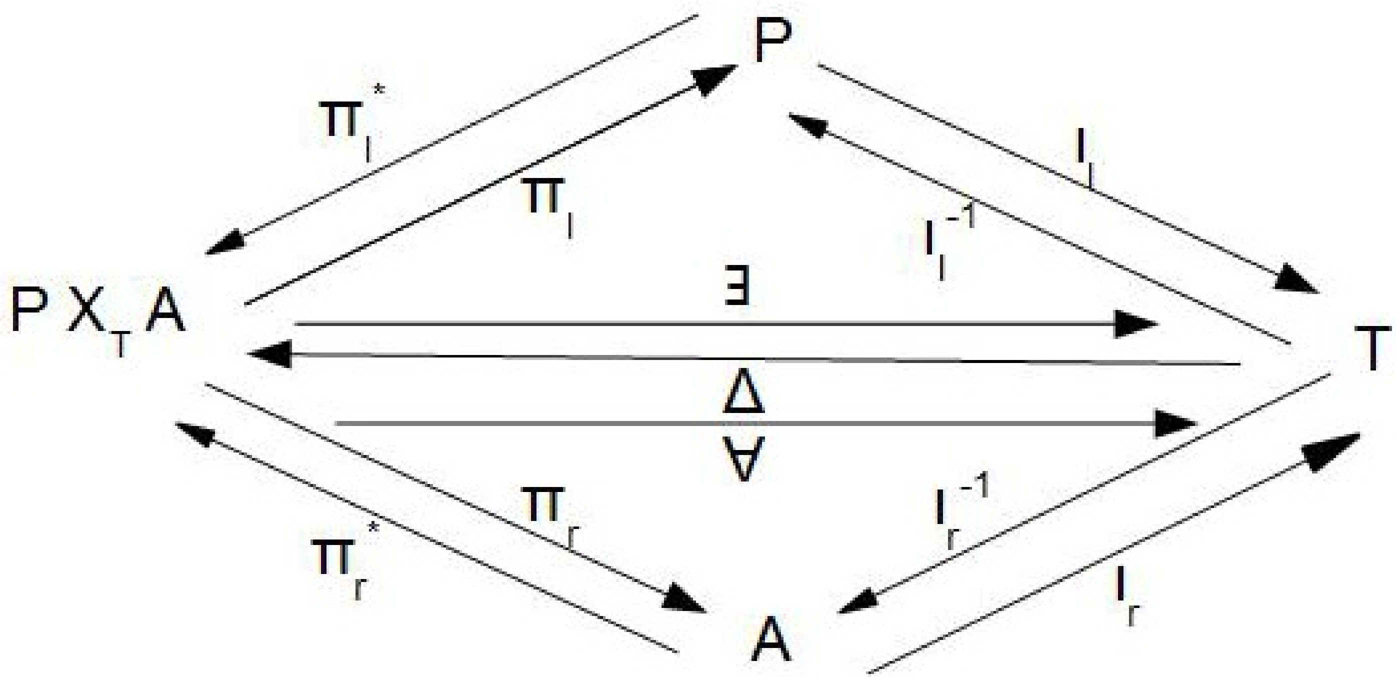


Figure 13: Internal Structure of Categories: a) The Pullback in **S**. S_X is id $\times_{S^+}$ id, S^+ is name $+_{\mathrm{id}}$ address. b) The Pullback in **M**. M_X is no $\times_{M^+}$ no, M^+ is title $+_{\mathrm{no}}$ grade, c) The Pullback in **R**. R_X is id $\times_{O^+}$ no, R^+ is mark $+_{\mathrm{id+no}}$ decision.

Each node in the pullback for Student over Marks in context of Result
is itself a pullback, giving a recursive structure

Pullback - Single Relationship:
Bank Transactions by Procedure and Account

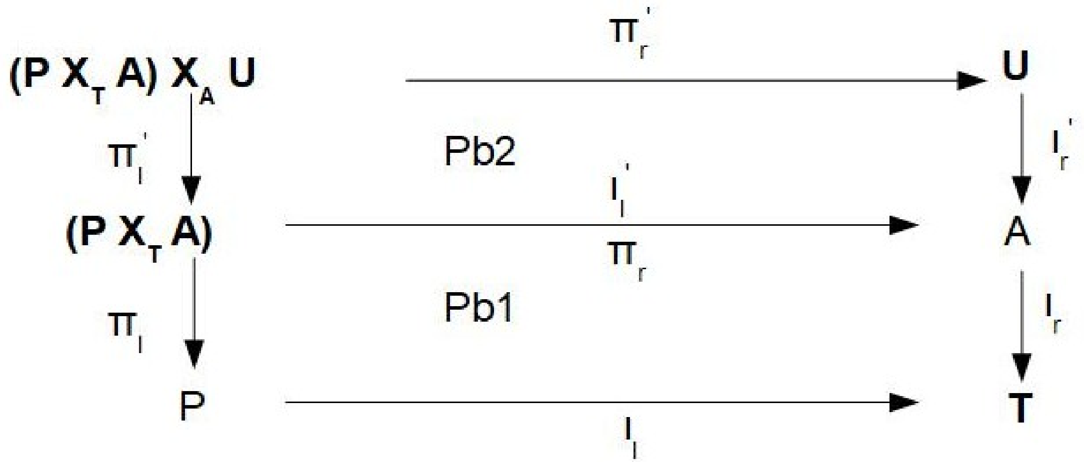


Pullback - Single Relationship Details

- $P \ X_T \ A$ (Procedure $X_{Transaction}$ Account)
  - Procedure is type of transaction: e.g. standing order, direct debit, ATM cash withdrawal
  - Account can belong to many users
  - Transaction is item for transfer of funds according to ACID requirements
- P, A,T are categories, with internal pullback structure, giving recursive pullbacks

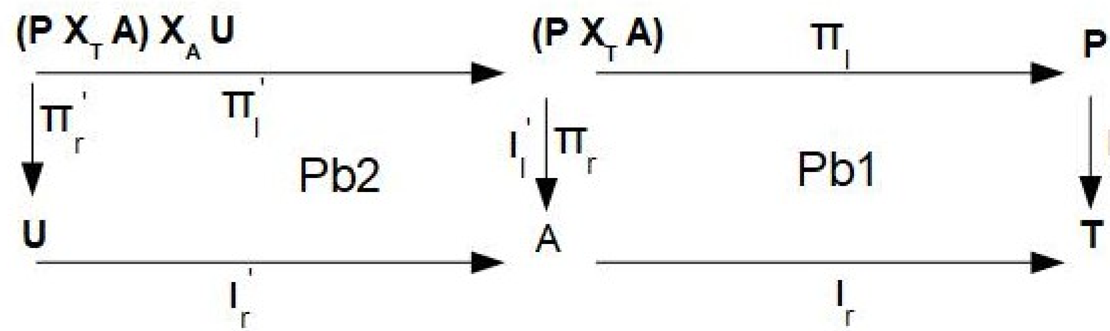
Pullback - Two Pasted Relationships: Bank Transactions bv User/Account

Three Pullbacks Pb1, Pb2, Pb2 X Pb1



U is user
A is account
T is transaction

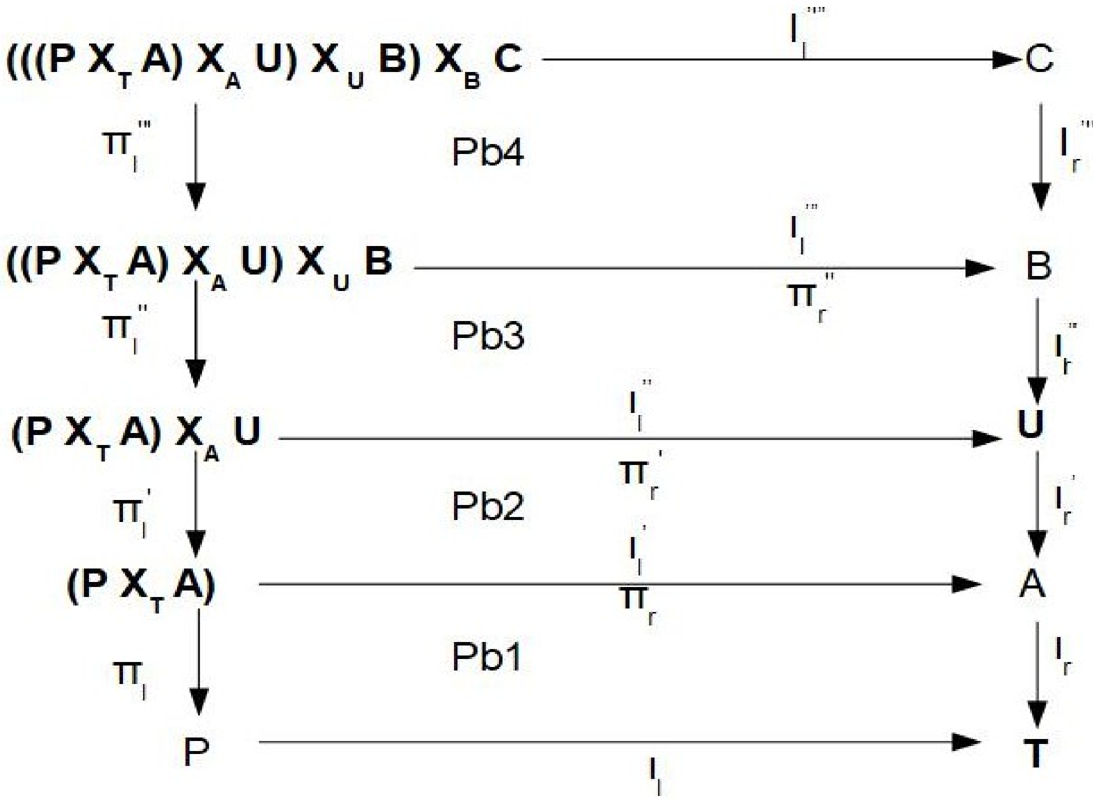
Usually written in horizontal (landscape) form. Vertical layout enables deep nested structures to be represented more readily

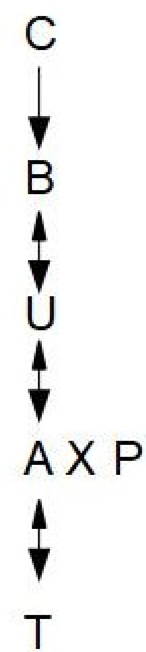


Pasting condition for Pb2 X Pb1: $\iota_l' = \pi_r$ after Freyd's Pasting Lemma

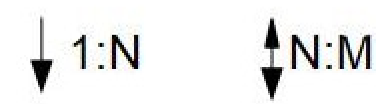
For our purposes, a pasted pullback is only a valid pullback if all inner and outer diagrams are pullbacks
Pasting is associative (order of evaluation is immaterial) but not commutative (relationship A:B 1:N is not same as A:B N:1)

Pullback – x10 Natural
Bank Account Transactions





N:M and 1:N are handled by same pullback structure



C company, B branch, U user, A account, P procedure, T transaction

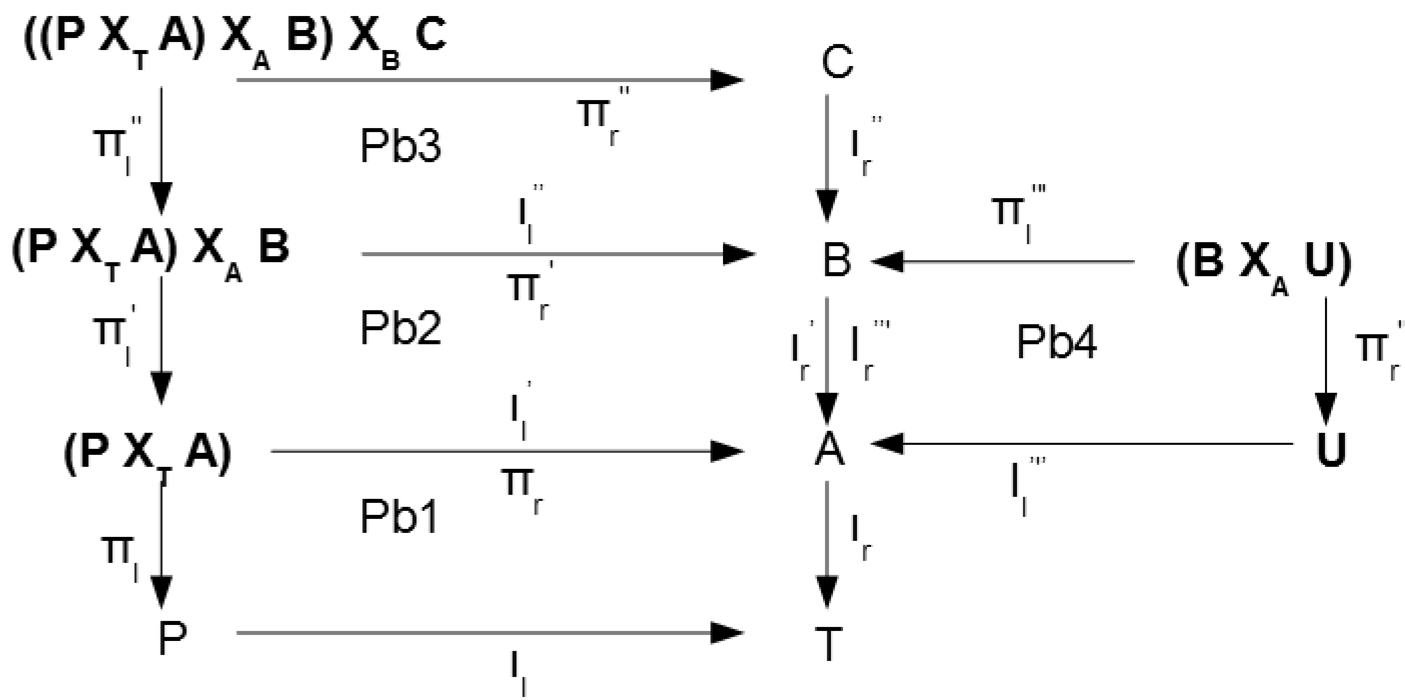
10 pullbacks: Pb1, Pb2, Pb3, Pb4
Pb2 X Pb1, Pb3 X Pb2, Pb4 X Pb3
Pb3 X Pb2 X Pb1, Pb4 X Pb3 X Pb2
Pb4 X Pb3 X Pb2 X Pb1

For our purposes, a pasted pullback is only a valid pullback if all inner and outer diagrams are pullbacks

Invalid Pullback

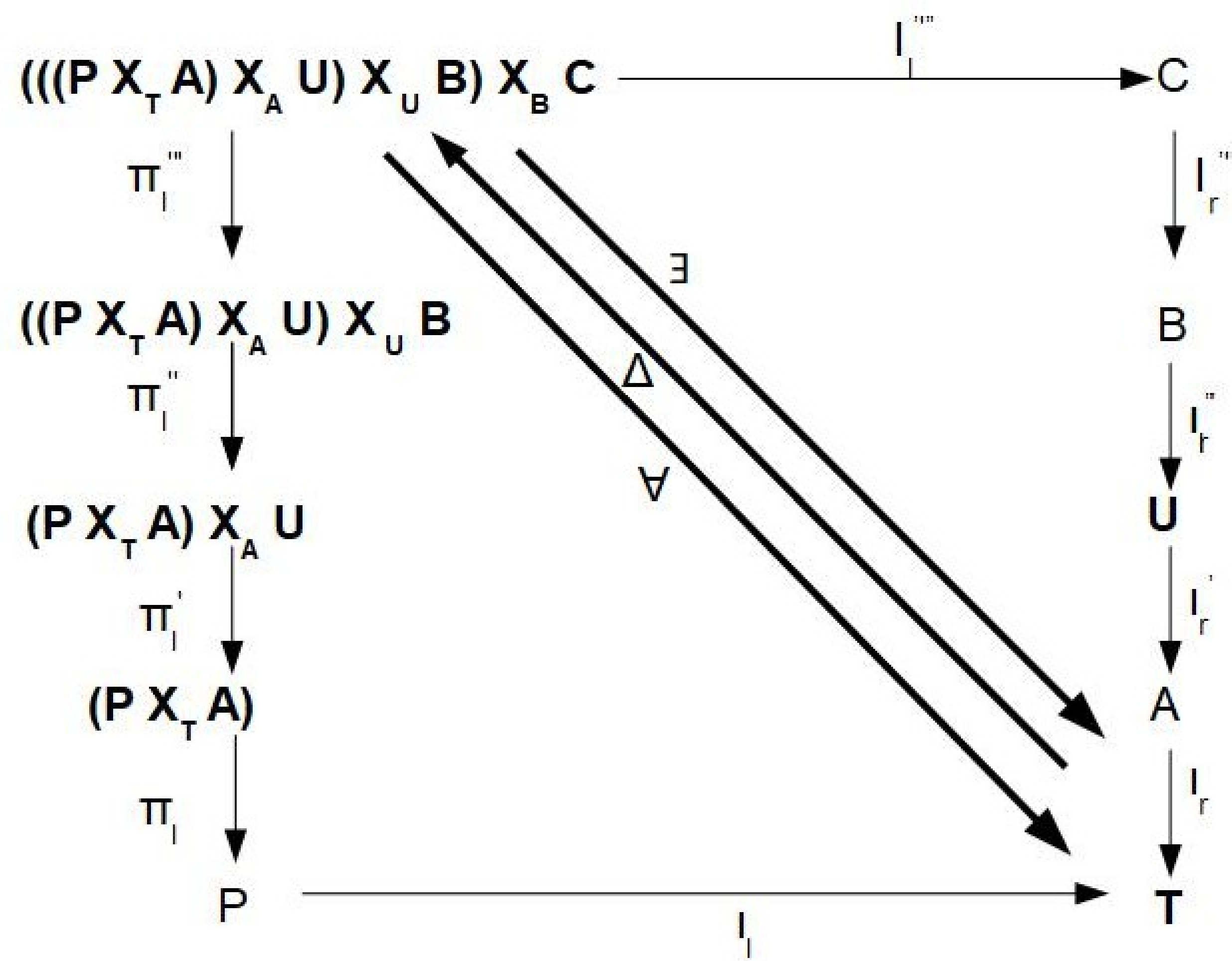
Invalid as
not all squares
are pullbacks

For instance
Pb4 X Pb2 is not
a pullback



Adjointness Holds for all Pullbacks

∃ ┤ Δ ┤ ∀ for this outer pullback and all other 9 inner pullbacks



Pasting Pullbacks – Discussion 1

- All diagrams commute
- All diagrams, inner or outer, are pullbacks
  - In pure maths, the condition is relaxed a little
    - Not appropriate for applied
- Structure is recursive
  - A pullback node may be a pullback structure in its own right
  - No limit to recursion

Pasting Pullbacks – Discussion 2

- Pasting condition appears to be:
  - $\iota_l' = \pi_r$ (left-inclusion of outer square = right-projection of inner square)
  - Discussed further later

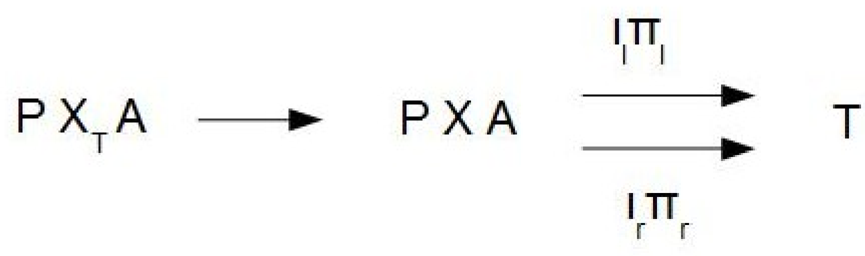
Pasting Pullbacks – Discussion 3

- Pasted structure
  - is a Cartesian Closed Category (CCC) with products, terminal object and exponentials
  - is a topos as a CCC with subobject classifier and internal Heyting Logic
- The subobject classifier provides an internal query language

The Pasting Condition 1

- $\iota_l' = \pi_r$ (left-inclusion of outer square = right-projection of inner square
  - Looks rather set theoretic
- But any pullback can be represented as an equalizer (ncatlab)

Equalizer for Pullback



Maps relation onto product onto context via 2 paths through pullback

The Pasting Condition 2

Similarly for a pasted pullback, the equaliser is

$$(P \times_T A) \times_A U \longrightarrow (P \times_T A) \times U \underset{I_r \pi_r \pi_r'}{\overset{I_l \pi_l \pi_l'}{\rightrightarrows}} A$$

Equals in sets is undefined as context is not defined

Equaliser in categories, as a limit, is fully defined up to natural isomorphism

External Process

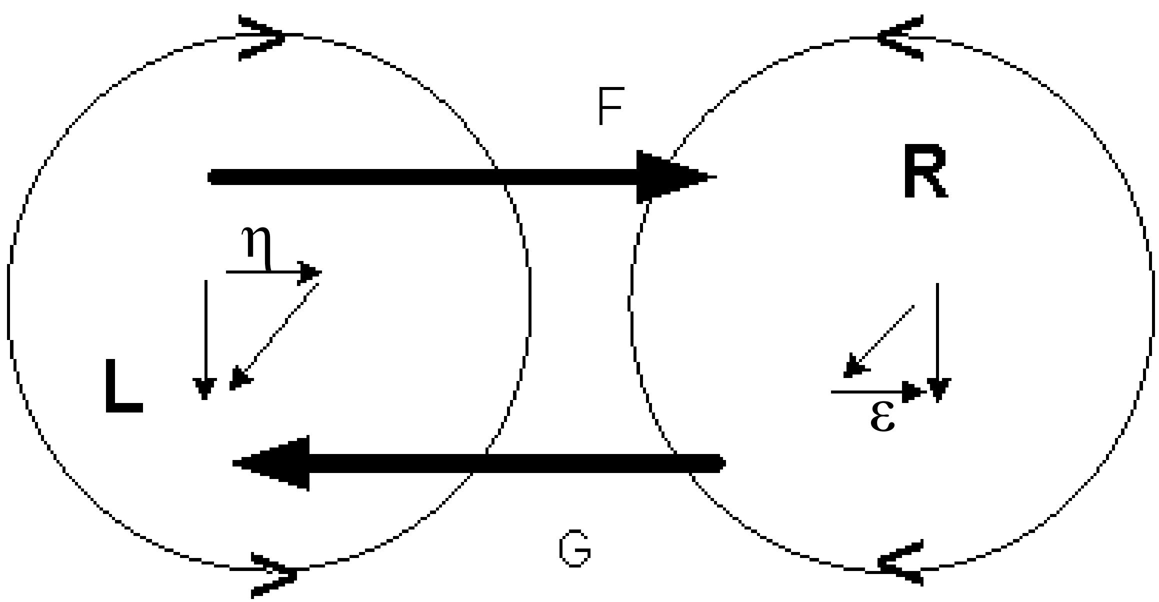
- Metaphysics (Whitehead)
- Transaction (universe, information system)
- Activity
  - Can be very complex but the whole is viewed as atomic – binary outcome – succeed or fail
  - Before and after states must be consistent in terms of rules
  - Intermediate results are not revealed to others
  - Results persist after end

Transaction in Category Theory

- In earlier work (ANPA 2010) we used adjointness to represent a transaction
  - Employing multiple cycles to capture ACID
- The aim now is to abstract this work using the monad, which we earlier described as the way forward
- The monad is an extension of the monoid to multiple levels
  - Monoid: $M \times M \rightarrow M$, $1 \rightarrow M$ (binary multiplication, unit)

Multiple 'Cycles' to represent adjointness

- Three ‘cycles’ GFGFGF:
  - Assessing unit η in L and counit ε in R to ensure overall consistency
  - 'Cycles' are performed simultaneously (a snap, not each cycle in turn)



F -| G

$\eta: 1_L \rightarrow GF(L)$ $\varepsilon: FG(R) \rightarrow 1_R$

Promising Technique - Monad

- The monad is used in pure mathematics for representing process
  - Has 3 'cycles' of iteration to give consistency
- The monad is also used in functional programming to formulate the process in an abstract data-type
  - In the Haskell language the monad is a first-class construction
    - Haskell B Curry transformed functions through currying in the λ-calculus
    - The Blockchain transaction system for Bitcoin and more recently other finance houses uses monads via Haskell
      - Reason quoted is it's a simple, reliable and clean technique

Monad/Comonad Overview

- Functionality:
  - Monad
    - $T^3 \rightarrow T^2 \rightarrow T$ (multiplication)
    - 3 'cycles' of T, looking back
  - Comonad (dual of monad)
    - $S \rightarrow S^2 \rightarrow S^3$ (comultiplication)
    - 3 'cycles' of S, looking forward
- Objects:
  - An endofunctor on a category X

Using the Monad Approach

- A monad is a 4-cell <1,2,3,4>
  - 1 is a category X
  - 2 is an endofunctor (T: X → X, functor with same source and target)
  - 3 is the unit of adjunction $\eta: 1_X \rightarrow T$ (change, looking forward)
  - 4 is the multiplication μ: T X T → T (change, looking back)
- A monad is therefore <X, T, η, μ> (or <T, η, μ> or <T, η, GεF> or in usage T)

The Comonad

- The dual of the monad
- A comonad is a 4-cell <1,2,3,4>
  - 1 is a category X
  - 2 is an endofunctor (S: X → X, functor with same source and target, S is dual of T)
  - 3 is the counit of adjunction ε: S → 1_X (change, looking back)
  - 4 is the comultiplication δ: S → S X S (change, looking forward)
- A comonad is therefore <X, S, ε, δ> (or <S, ε, δ> or <S, ε, FηG> or in usage S)

Monad is often based on an adjunction

- The transaction involves GF, a pair of adjoint functors F -| G
  - F: X → Y
  - G: Y → X
- GF is an endofunctor as category X is both source and target
- So T is GF (for monad)
- And S is FG (for comonad)

Process: Operating on a Topos

- The operation is simple:
  - T: E → E'
    - where T is the monad <GF, η, GεF> in E, E', the topos, with input and output types the same
- The extension (data values) will vary but the intension (definition of type) remains the same
- Closure is achieved as the type is preserved

Composability is the Key

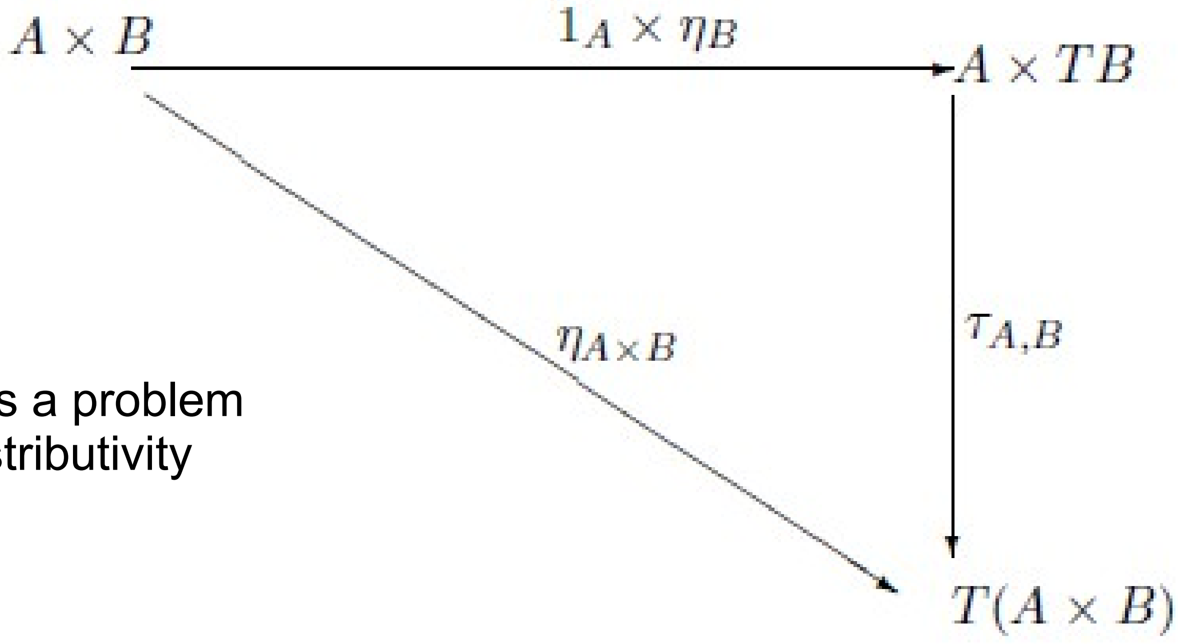
- Compose many monads together to give the power of adjointness over a whole wide-ranging application
- In banking (Bitcoin) the reliability obtained from composing processes over a wide-range of machines (distributed data recovery) justifies the move to Category Theory
- There is a problem though in EML Category Theory:
  - Monads do not compose naturally

Haskell and Monads

- Kleisli Category of a Monad
  - Transforms a monad into a monadic form more suitable for implementation in a functional language
    - Used in Haskell rather than the pure mathematics form of Mac Lane
- Strengthens the monad for composability
  - As in the Cartesian Monad, with products
- A practical application of the pure maths has exposed problems in the maths
- Solution has come from another pure mathematician Kleisli

Kleisli Lift

- Define a natural transformation:
  - $\tau_{A,B}$: A X TB → T (A X B) where A,B are objects in X and T is the monad such that the following diagram commutes



There is a problem
with distributivity
In EML

Summary of Progress/Look forward

- Topos has been established as data-type of choice
- Monad shows potential for processing the topos
- Advent of Haskell gives an experimental test-bed
- Next application area is music (Music as a Composition of Cartesian Monad over a Topos, ANPA 38, Hampshire, UK, August 2017)