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THE TOPOS OF CATEGORY THEORY AND REALITY

The rational clarity that comes with a formal approach has greatly advanced physics and technology in first order applications. A scientific approach to human activities from biology and medicine across to language, the arts, law, religion and philosophy, etc, on the other hand has in the past been limited to classificatory procedures. Higher order formalism is now available for all disciplines with categories beyond set theory. By the representation of process with the 'arrow' of category theory it is possible to develop by natural reasoning the notion of the world as a closed Cartesian category with a structure of adjointness between universal limits. This is a tutorial on fundamental concepts of category theory for those in any discipline and requiring no prior expertise in classical mathematics.

***Key words:* ....add key words**
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1 Process

Although its roots go back a further twenty years, the Alternative Natural Philosophy Association (ANPA) has from its first meeting in 1979 given preference to rigorous formal argument. `Disciplined thought' is essential with alternative methods. Without pre-existing agreed defined terms, ANPA would otherwise have no *modus operandi*. For there has to be some common ground of reasoning. Process seems to arise naturally as both a consequence and a catalyst in the ANPA context. A continuing example is the process basis for the fine structure constant [5].

A fundamental structural significance in the world is the way the local connects into the global such as in the McLuhan Global Village where everything is connected [20]. The temporal analysis is the distinction between stationary and the non-stationary. Philosophically this global/local distinction is not at all new. It is at the root of Zeno's paradox of the arrow's dynamic flight consisting only of static positions.

The noun 'process' or the participle 'processing' commonly describe an act of transforming an existing object by some procedure to another form as in a manufacturing or business administration procedure. Wikipedia deals with its entry for PROCESS in up to 40 different fields of knowledge, including philosophy, science, engineering, computing, chemistry, biology, law, business and even the 'process haircut' [32]. There are variations in the meaning of the word depending on context. For instance in business, process describes activities or tasks that produce a specific service or product for customers. Interestingly Wikipedia does not include physics in its lists of fields for process.

The whole subject of cybernetics can be viewed as a process operating in nature as in Wiener's definition [30] involving comparison of communication in the animal and the machine [24]. Process describes the way that both animals (biological systems) and machines (non-biological or "artificial" systems) can operate according to cybernetic principles. This was an explicit recognition that both living and non-living systems can have purpose. Wiener considered that systems theory seeks to deal with the local/global divide [24], treating systems as equivalent to process but the latter is the higher form. The early specification of the working of the brain in cybernetics by Ashby [2] amounts to the concept of process but it was von Bertalanffy of the early founders of cybernetics that explicitly related the latter to process [6, 7].

Most writers trace process to the 'all is flux' of Heraclites in contrast to Parmenides, who is more usually associated with a static view. However, process is more than flux and also subsumes permanence. It is rather the Heraclites' *logos* which was taken up by the Greeks of Alexandria and the Judeo-Christian tradition to identify *logos* with God and the second person in the Trinity. The whole theory of evolution is process too but one where the origin of species does not unfold in a linear fashion. Evolution appears a foundational natural process encompassing both emergence and change. Ordering is adjointness and includes both static and dynamic aspects. It is a paradox that process includes invariance[1] which describes no change under a transformation. Indeed scale invariance turns out to be an important phenomenon of process and a relevant aspect to ANPA because of the interest in dimensionless universal constants such as the scale invariant fine structure constant. Fractal patterns arising from scale invariant physics are studied piecemeal with use of special sets like the Mandelbrot, Julia etc. However general methods are restricted because a set cannot be a member of itself in the way that a reflective subcategory can have itself as an object. Information systems like the web also exhibit properties of scale invariance but we do not have space here to pursue this aspect of process which arises in exponential categories.

There is always the problem of where to begin. That statement may be formally expressed as *a pre-order of categories* or just as well as *a category of preorders* for both lack beginning and ending. However within process we can but focus on the *category of reality* in the sense of the category where objects and relationships between objects exist to make up the physical world. This is metaphysical process and the Universe is an instantiation of process but the World is even greater than the physical Universe consisting of all the relations between physical

[1] The subject of invariance was mainly developed in the 19th century by Arthur Cayley. Saunders Mac Lane [21] traces the early origins of category theory to Cayley.

entities and all the relations between those relations. Physical relations connect directly from higher-order relations. This is treated bottom-up but because of the holistic nature of process it is driven top-down. A topical example is the recent realisation of how subjective human behaviour affects the objective syntax of world economies. Current practical examples of applied recursion across levels is deduplication in structured data storage [15] or functional DNA nanostructures that can be integrated into larger structures as miniature circuit boards in bioengineering [26].

In this sense the *World* is greater than the physical Universe of cosmology. There is a unique arrow from the source of the World to every object in it and a unique resultant arrow between any pair of objects as in Figure 1.

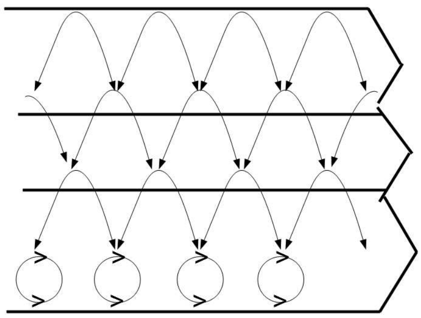

Fig. 1: A Schematic World-Universe Relationship. The lowest horizontal arrow is a category consisting of a row of notional elementary objects connected by a row of vertical arrows which are themselves connected by a higher row which are in turn connected by a yet higher row.

For we are concerned with the higher order of relations between physical objects and the relations between those relations which together with the physical objects of the Universe make up the World. This then embraces the whole of human affairs and activity including the arbitrary disciplines of philosophy and theology. Existence in categories is identifiable with the object which as we shall see is the condition known categorically as Cartesian. Ordering is adjointness and includes both static and dynamic aspects.

This empirical knowledge that every entity that exists is related to every other object that exists is no more than a definition of the Universe to include everything naturally accessible. This provides a unique direct arrow between any pair of objects that is the composition of all possible arrows between them. This is the structure given the label preorder. Figure 3 presents a two-dimensional representation for the context of the objects *C* and *A* of a preorder. There is but one unique arrow between any pair of objects in a preorder and that arrow as the figure shows is the limit of all other possible arrows whether directly between the pair or indirectly via any other pair in the preorder. We cannot assume any orientation for it or even presume the concept of a dimension. It is possibly easier to imagine than to draw, although our common perception wants us to imagine it in three dimensions or possibly in higher order algebraic or geometric dimensions but lies easily in higher-order geometric dimensions. A process preorder does not exist in any space whether algebraic or geometric. Rather it should be space free. This is the quantum world. However the effect between entities is mutual and the arrow is therefore two-way but not symmetrical because the opposing directions give rise to a natural parity in their mutuality. This is the ultimate reality of the quantum world. Whether it is the quantum or the physical world that is true reality seems just a matter of personal preference.

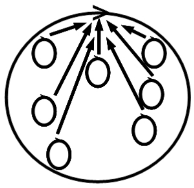

Fig. 2: Terminal Object. A category with a terminal object has an arrow from every object to it. In preorders this arrow is unique as in Figure 3

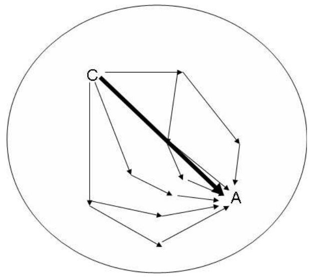

Fig.3: The unique arrow from *C* to *A* as a limit of other arrows in a unidirectional Preorder. The co-limit sums over all others

Newtonian physics treated the universe as some container either rectilinear or spherical but embedded in time. Such a structure is representable, for example by Yoneda or Curry techniques[2], to first-order as a number. This is the classical model which can be verified by measurement in first-order predicate logic because as Gödel has shown first-order predicate logic is complete for a closed world. However Gödel has also shown that such a logic is not complete for an open world and any model based on number and relying on axioms is not complete whether open or closed [10]. This effectively sets a limit to Wigner's 'unreasonable effectiveness of mathematics in the natural sciences' [31].

2 Metaphysics

If we want to identify a category with reality, existence requires designation of one object as the terminal object, as shown in Figure 2. This is the condition known as 'Cartesian'. It is also possible to designate another as the source of the process as initial object. This is the condition known as 'co-Cartesian' but is not a necessary and sufficient condition and may therefore result in over-specification and a too constrained system. There is a free functor mapping from the preorder on to any of its partial orders. It is natural to identify the terminal object with the covariant identity functor. If the initial object exists it would exist as the contravariant identity functor of the category. Nevertheless although these are arbitrary terms the use of the labels 'terminal' and 'initial' imports an interpretation and requires the existence of some axiom of choice, which is an axiom/assumption of set theory. The ANPA *Statement of Purpose*[3] Clause 1 states that 'The

[2] See [3] at pages 118 and 190 respectively
[3] as regularly published in its Proceedings including in these proceedings for ANPA 31.

primary purpose of the Association is to consider coherent models based on a minimal number of assumptions. Here we are raising the stakes from models to metaphysics but nevertheless attempting to keep to a minimum of assumptions[4]. The *Statement* might be better expressed as 'a minimum of assumption'. Here we try to make no assumption at all beyond that the World exists. We try to keep open issues about terms such as 'terminal' and 'initial' because they may be related to what cosmologists currently tell us about the fabric of the physical Universe consisting mainly of dark matter and dark energy with only 4% in familiar forms.

ANPA is mainly concerned with fundamentals at the frontiers rather than incremental advances within existing knowledge. But how do the general and the particular relate within the structure of the world? Any formal description needs to be able to combine both the global and the local. This is possible with natural categories by substituting metaphysical process in the interpretation of Whitehead's later *Process and Reality* [28] for that in his earlier *Principia* [27], which was the starting point for the traditional finitary category theory of Eilenberg and Mac Lane [21]. It is the difference between a metaphysics and modelling which are separated by two levels as in Figure 4 (diagram 16 in [25]). Metaphysics is one level up from reality in human perception while models are one level down. The limitations of modelling reality can be seen in information systems where there is a need to represent the world on computers. Problems are evident in database methods like ACID [11, 23] and in Codd's pure relational model [8]. In database design, data normalisation is used to attempt to match the logical data structures to the physical world. This method of design has a number of unsatisfactory features. Firstly it is difficult to enforce the laws of the physical world in the operational database and secondly the theoretical underpinning, based on set theory, is not natural because of the problem of representing arrows across multiple levels as functions.

3 Finitary categories model natural (metaphysical) categories

Whitehead developed his theory *Process and Reality* in what he terms speculative metaphysical categories. These are in great contrast to the formal principles he enunciated with Bertrand Russell in *Principia Mathematica* and Whitehead devotes Chapter 1 of his later work ([28] pp.4-26; [29] pp.3-17) to explaining in a general philosophical context why they had to be speculative. For the second half of the last century has seen substantial advances in the development within finitary mathematics of formal categories based on the concept of the arrow and initiated by Eilenberg and Mac Lane [21]. The phrase 'finitary mathematics' is a term first coined by the mathematician David Hilbert[5] and effectively describes the whole mainstream of twentieth century mathematics built up on a system of proofs in set theory and number from incompletely specified axioms. The adjective finitary is itself a little misleading as finitary mathematics includes topics like infinity and transfinite numbers as these are modelled on the finite concept of number.

Nevertheless it is possible to ascend the staircase in Figure 4 from categories as mathematical models to metaphysical categories and extend that ladder to categories that are no longer speculative but which can now be made formal thanks to the work of Eilenberg, Mac Lane and a large number of pure mathematicians world-wide who have refined and extended their original interpretation of the humble arrow based only on the four properties:

1. a morphism from domain to co-domain
2. identity from an indistinguishable domain and co-domain
3. associativity
4. composition.

There are two distinctions important for process that we need to draw. One is between metaphysical categories and finitary categories in respect of the use of number in physics; the other is between sets and either types of categories in respect of the representation of intension and extension. We will first consider the natural numbers then look at intension and extension as an intrinsic property of parity to be found in adjointness.

4 The finitary category of the natural number

Because it relies heavily on experiment, physics as a discipline has become identified with measurement and number as its prime conceptual tool. Consequently it has become very bound up with sets which equate to number. However it is an assumption that qualities and quantities are representable as number. The physics and the mathematics have become merged so that space is a complex number whether it is Newton's Universe as a container or the infinite Hilbert space of quantum mechanics. These it should not be forgotten are just numbers. This is fine to the extent of first order models for which Gödel (as mentioned above) has shown to be consistent but it is not sufficient for open or other higher order systems for Gödel has shown these to be neither consistent nor decidable when relying on axioms of sets or number. This applies as much to the use of statistical methods as the interpretation of measurement. It may be possible to reduce any problem to first order but any conclusions will then be subject to the assumptions in the reduction. This is particularly insidious in treating open systems as closed. However openness is not just bound up with the concept of order for it contains a deeper logical strand of constructivism as associated with the intuitionism of Brouwer. Boolean logic suffices for a closed system but an open system requires the logic of Heyting (See Figure 9 below).

Metaphysical categories have therefore no natural concept of number. Finitary categories as a model relying on the concept of sets has consequently to introduce the concept of number[6]. This is achieved by postulating a Natural Number Object with a recursive definition on arrows comparable to recursive functions generating the set of natural numbers. This requires importing some undefined successor function. While this may be natural in

[4] There is some philosophical difficulty here with ANPA`s `minimal number of assumptions' when dealing with supposition because the number is not necessarily a measure of quantity or quality.

[5] according to Feferman [10] Hilbert never defined finitary mathematics and it collapsed at its foundations under the weight of Gödel for the reasons mentioned above.

[6] First carried out by Lawvere [17] and now to be found in standard category theory texts, such as [3] at p 177.

mathematics it is not natural in physics where systems are open either externally or internally. An obvious example is radioactivity where atoms decay according to some preorder and it is not therefore possible to identify a successor before the event of decay. Of course it was to explain such events that the notion of randomness was invented but this is normally dealt with by some theory of statistical probability which leads back to the concept of number and does not provide an exact solution. This lack of a predefined successor is a feature of all open systems and a chief cause of problems of interoperability in global systems.

Open physics lacks a concept of number and this questions the use of finitary models in physics. The existence of multiverses must surely be the largest incarnation of the number concept. The Panel 1 lists nine current possible theories recently identified by Greene [12]. These can also be compared with Barrow's views of multiverses [4].

Panel 1 : MULTIVERSES - Current Possible Multiverses recently identified by Greene [12]

1. Infinite space may contain a number (possibly an infinity) of universes that may lie beyond our sight.
2. Uncountable other universes with different characteristics may have been created with ours during a fleeting period of superfast accelerating expansion.
3. String theory suggests our universe is one of many 4-dimensional '*brane worlds*' floating in a higher-dimensional space-time.
4. A simple cycle of universes with variations in physical laws as possible in string theory.
5. More complex versions of cyclic universes.
6. Quantum mechanics allows/requires many worlds to exist in parallel formed by a branching of the wave function.
7. The universe is a holographic projection.
8. We are just one of a set of artificial universes created in simulation on a superadvanced computer.
9. The philosophical necessity that every possible universe must be realised somewhere.

It is instructive to review Greene's list from the process perspective. The list does not claim to be exhaustive and is an example of undecidability demonstrating how the use of number leads to degeneracy with many possible forms. This degeneracy is well borne out in the thorough examination of n-categories carried out by Leinster [19]. It may well be a comparable defect in string theory that allows variations in physical laws. In process categories physical laws arise from properties of adjointness whose *bonum esse* is uniqueness. Furthermore about half the items in the list depend on some idea of infinity. But infinity belongs to mathematics, not to physics. It was David Hilbert the proponent of finitary mathematics who with the paradox of his Hotel Infinity recognised that infinity is always beyond reach and therefore cannot plausibly exist in physical reality. Infinity in finitary mathematics seems no more than a model of repleteness[7] under the free functor in process categories. The last item that postulates every possible universe is also derived from probability theory applied to infinity. That too fails at the Gödel hurdle of 'number'.

As anthropocentric variants on our universe with complicated theories reminiscent of epicycles, multiverses bear an almost Ptolemaic resemblance. The super-advanced computer is a science-fiction vision of current commercial computers. They have not been thought through with respect to quantum computation nor any general attention paid to boundary conditions nor to the relativistic nature of time which Whitehead would carefully respect [28].

5 Logical structure of World representation as adjointness

In terms of natural categories, process is adjointness. This is the formal metaphysics of real existence such that every physical entity in the Universe affects every other. There is at the most a single pair of arrows in opposite directions between any pair of objects. These are limits of all the possible paths around the Universe between any given pair. This limit reduces to a single function as an abstraction in lambda calculus or as a resultant in vector analysis (for first order models lose the resolution of the contravariant pair). There are four levels involving three interfaces. The uppermost level is the intension and the lowest is the extension corresponding respectively to the global and the local. The intermediate interface connects intension and extension, that is snaps the local into the global for all time and space. Any set-theoretic approach finds this latter mechanism, which is essential to all studies of globalisation and interoperability, very difficult if not impossible as recognised by Russell's paradox.

Nevertheless in finitary categories the mathematics of adjointness has been developed in this concept termed a Cartesian closed category, derived as an abstraction of the Cartesian product but this description from historic origins may by its simplicity mislead as to its great power and content. The finitary approach is to distinguish the two properties of Cartesian closed and locally Cartesian closed but in process categories it is that natural distinction between intension and extension. **This paper provides an introduction to that formal description of the mathematical structure of the World as found in nature.**

To the global/local distinction must be added the stationary against the non-stationary. Both the static and the dynamic are formally representable and accessible in the logic of natural categories. Process relates not just to the non-stationary but subsumes both the static and the dynamic. One is contained in the other but which way round? Such problems, like Zeno's paradox of the arrow's dynamic flight consisting of only static positions are avoided in the 17th century French logic school of the Port Royal [1] (harking back to Aristotle's first and second intentions) by distinguishing the intension from the extension. Aristotle referred to them as first and second intentions. Because of their extended meaning these terms were recognised in the subject of logic by retaining the older spelling with an "s" rather than a "t". When the old subject of logic was superseded around 1900 by symbolic logic based on set theory, the intension/extension relationship became rather lost until the development of computer programming revived it with the need for rigorous typing.

[7] Johnstone ([16] at p.3) defines the condition of repleteness as "that any object of the ambient category isomorphic to one in the subcategory is itself in the subcategory".

The intension-extension relationship is recursive; thus in the diagram of Figure 4 metaphysics is the intension for reality as its extension and reality itself becomes the intension for models as possible extensions. In the natural categories of metaphysics process is adjointness. This is no more than the formal metaphysics of real existence that every physical entity in the Universe affects every other. There is at the most a single pair of arrows in opposite directions between any pair of objects. These are limits of all the possible paths around the Universe between any given pair. This limit is that of the preorder in Figure 3. Mathematical categories other than the Cartesian closed are possible but process categories being derived from physics only recognise the existence of Cartesian closed categories which has the property of adjointness. Every object is the domain of a covariant arrow and the co-domain of a contravariant arrow. This recursive structure of intension/extension applies at any level but is best studied between a pair of categories (identity functors 1_F and 1_G) where adjointness of the pair of arrows (**F** and **G**, contravariant to one another) induce a monad consisting of a triple < T, η , μ > and a co-monad consisting of the co-triple < S, ε , δ >. Figure 5 shows the adjointness between the categories, intension and extension.

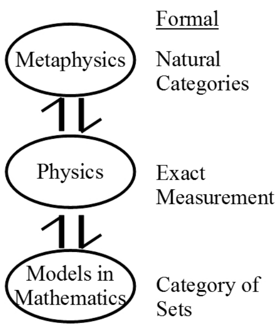

Fig. 4: The Staircase of the World from Metaphysics to Models

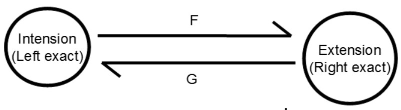

Fig. 5: Adjointness F $\dashv$ G

Each arrow has a dual role. *F* is the contingent arrow of intension and the determinant arrow of extension while *G* is the contingent arrow of extension and the determinant arrow of intension. *T* is just the composition *GF* and *S* the composition *FG*. Each of these compositions may be compared in Figure 6 at the next level up with the contribution they make to their respective identity functors by means of the creative unit of adjunction η : $1_F \rightarrow GF$; and the qualitative co-unit of adjunction ε : $FG \rightarrow 1_G$. Comparison at the even higher level of order is provided by the unit of potentiality $\mu : T^2 \rightarrow T$; and its co-unit $\delta : S \rightarrow S^2$. There are special cases of the latter two which may be interpreted [25] as in the 'dimension of time' with the unit of anticipation where potentiality is by hindsight and the co-unit of anticipation by foresight. Although there are never more than two basic adjoint functors $F \dashv G$, the combined composition of their two compositions *T* and *S* may be resolved into the three basic functors of Figure 7 to be found in standard category theory texts, where Σ is the existential qualifier, Π the universal quantifier and Δ the stability diagonal pullback functor. The interplay of left and right adjointness with left and right exactness is a little subtle [13] and can be better understood in the exploded diagram of Figure 8 which is repeated in Figure 9 to show an exploded view of the natural intuisionistic logical structure of the Cartesian closed category.

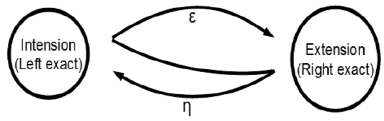

Fig. 6: Adjointness expressed with natural transformations η and ε

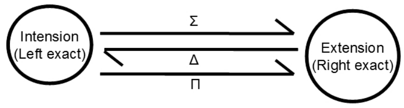

Fig. 7: Adjointness Σ $\dashv$ Δ $\dashv$ Π

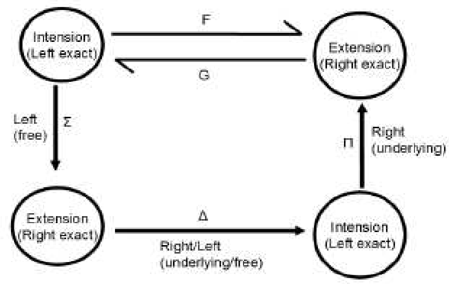

Fig. 8: Explosion Σ $\dashv$ Δ $\dashv$ Π of the Arrow Functors of Adjointness F $\dashv$ G

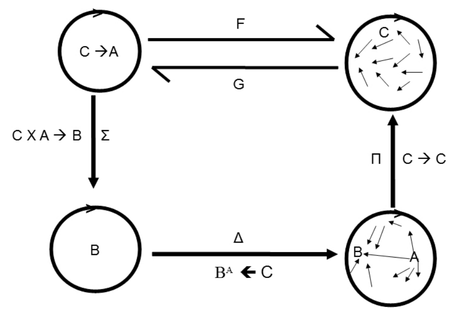

Fig 9: Intuisionistic structure of the Cartesian closed category: Exploded view of Heyting logic.

6 The natural World structure as a Cartesian closed category

Relationships in nature are therefore all explicable in process categories with this single concept of adjointness [18] that consists only of a pair of contravariant arrows inducing a monad. In finitary categories the mathematics of adjointness has been developed in what is termed a Cartesian Closed Category, derived as an abstraction of the Cartesian product but this description from historic origins may by its simplicity mislead as to its great power and content. The finitary approach is to distinguish the two properties of Cartesian closed and locally Cartesian closed but in process categories it is that natural distinction between intension and extension that provides a formal description of the mathematical structure of the World as found in nature. It is the simple principle that everything in the world is related to everything else in the world that provides the formal structure of the relationship relevant to any scientific study or technological application requiring an understanding of these relationships.

An early example is the representation of information in computers that needed some implementable model of real-world relationships. Some variation of the hierarchical was possibly the most common structure attempted in different knowledge systems. But the most successful measured by the volume of commercial transactions was by far the simple relational model based on lists or tables manipulated as sets embodying an intension/extension relationship.

The Cartesian closed category (CCC) is a fundamental category of category theory. Its features and their definitions are to be found in its standard textbooks but most if not all come from the stationary viewpoint of set theory, not from process. That set theory itself does not rest on unequivocal foundations may raise few problems in pure mathematics where axioms may be defined at will and may well be adequate too in applied mathematics to a first order. However, many problems requiring mathematical solutions today arise in more complex situations. Transactions in information systems [22] are a case in point as of the nature of process. Thus a common approach in databases [9] is to adopt the principles under the acronym ACID stating the requirements for Atomicity, Consistency, Isolation and Durability. The aim is to ensure that a transaction involving a series of operations is indivisible, enforces all rules, provides results only on termination and guarantees to hold the results under any circumstances. The transaction concept has been implemented efficiently on many database systems but in information systems as a whole the idea lacks the abstraction needed for successful business modelling. The alternative approach in natural philosophy is that of process as explored in the 20th century [23].

While in the formal language of category theory the world may be described as 'Cartesian-closed', this term may give a false impression that it has a Cartesian coordinate system which is unfortunate but the phrase has arisen historically in that context because it embodies the fundamental concept of the Cartesian product. In fact it is much more than a simple product and these terms need to be examined further. For while natural categories and metaphysics provide us with a process structure for the world, we can only begin to investigate it here. Intension and extension alternate in a preorder, that is with an arbitrary beginning of an intension with an extension which itself becomes an intension of the next extension and so on as in Figure 10 [14].

7 The Topos: Archetype of Natural World

The archetype of the natural world is the topos, in its early days formally defined as a Cartesian closed category with subobject classifiers and informally as a generalised set. Johnstone in his preface to [16] lists thirteen alternative descriptions that have been applied to the topos (pp.viii &sq). Many of them like for instance "A topos is a generalised space" still carry hangovers from sets. We would recommend as an informal definition: "The category of categories of catgories". To some this may only confirm categories as "abstract nonsense" but it is accurate and makes explicit the recursion. The topos sums up all that we have said in this paper. It is the ultimate intension existing as an identity natural transformation in any extension given by the internal categories, subject to the locally Cartesian closed condition with the preorder structure and an intuitionistic logic that is the Heyting and which is more general than the Boolean. There is a unique arrow from the source of the World to every object in it and a unique limiting arrow between any pair of objects.

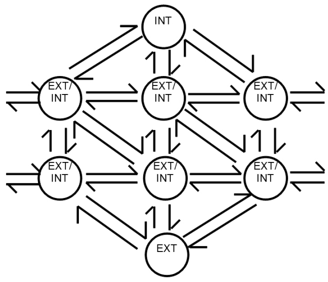

Fig. 10: Alternate Intension/Extension Pairs in Nature

To satisfy its holistic nature the World must emerge top-down. That is to say no more than that if the Big Bang happened it potentially contained everything that ever existed[8]. However it is easier to explain bottom-up by treating the role of the arrow as a natural expression of process with an identity arrow as intension and a distinguishable valued arrow for extension. However while in natural category theory the simplest identity arrow may be treated as an object, it is convenient to begin with a category of three composing objects as a generalisation of any possible category. This is shown in Figure 11 with the next higher identity arrow (the functor) composing extensional arrows between objects.

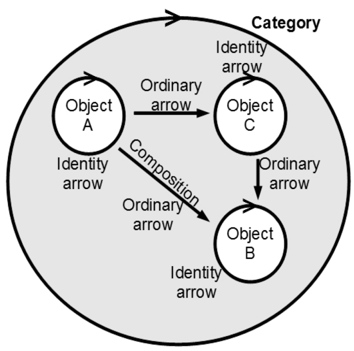

Fig. 11: A category consists of ordinary arrows composing between identity arrows as objects

The next higher identity arrow is the locally Cartesian closed natural transformation composing categories with ordinary functors as extensional arrows between categories as shown in Figure 12. The highest level arrow is also a natural transformation which composes structures of categories and functors. It is this identity natural transformation that constitutes the full Cartesian closed category of a topos as in Figure 13. However, the natural

[8] formally (μ; δ) in the description above for the monad/comonad.

arrow is double-headed as a composition of the adjoint functors but with a parity as previously discussed above. Although as just explained it may be easier to understand these diagrams bottom-up in the way that models are usually built-up, nevertheless process can only exist as a whole and the full diagram represents a natural occasion or "actual event" as first introduced by Whitehead [28]. From the long-term perspective of ANPA however the four-level Combinatorial Hierarchy based on the Frederick Construction is a binary model of the four-level metaphysical preorder of Process presented here.

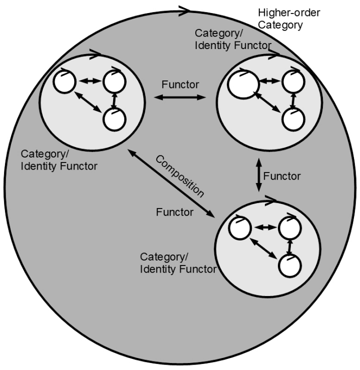

Fig. 12: Functors between Categories compose to form higher-order categories. A category is just an identity functor

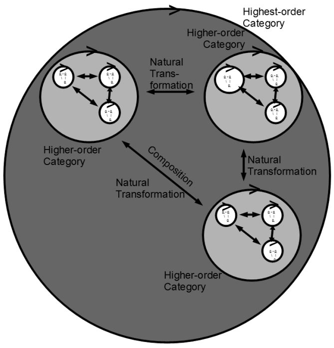

Fig. 13: Natural Transformations of Composing Functors themselves compose in the highest possible category, a Topos

The whole is just a recursive system with closure at four levels consisting of three open interfaces. Figure 13 shows the three interfaces for composing arrows (ordinary, functor, natural transformation) with the four levels (identity arrow, identity functor/category, higher-order identity functor/category, identity natural transformation/topos). The diagram shows well the natural recursive nature of the structure. It also demonstrates connectivity from any object to any other object. It is possible therefore, as shown in Figure 14, to get from any object A to any object B directly: $B = \theta A$, or indirectly with possible local variations through any other internal path: $\theta''\circ\theta' A = B$. This is a natural structure because it is obtained from simple induction applied to the notion of process without any assumptions. As a final comment it is interesting to compare briey the World as a topos with the long-term study by ANPA of the four-level Combinatorial Hierarchy based on the Frederick Construction as a binary model of the four level metaphysical preorder of process presented here. The subobject classifiers of any intension are the Boolean truth values (0,1) as the initial and final objects of a topos that is both Cartesian and co-Cartesian. The intension generates by process the possible extensions but is limited by scale invariance to four levels of three interfaces which the Frederick construction as a model predicts and correlates to a great precision with the fine structure constant.

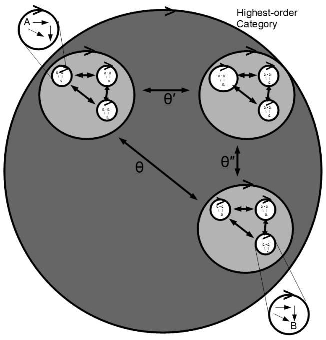

Fig. 14: A topos showing natural path from any object A to any object B

Appendix I: Finitary Approaches to Cartesian closed Categories

The relationship between natural categories and finitary category theory is symbiotic as part of the general mutual independence on one another of pure and applied mathematics. Natural categories being metaphysical are at the highest possible level and therefore lack a higher vantage point from which to view them. Finitary category theory on the other hand is a model relying mostly on the category of sets. Being finitary the subject can be advanced by a number of categorial proofs. Understanding categories on the other hand has only pure induction to guide by empirical reality through the natural metaphysics. This is an important example of the three-tier general scheme of metaphysics, physics and models of Figure 4. Because of the symbiosis between the pure and applied approach to formalism it is instructive to compare the traditional treatment of Cartesian closed categories in finitary category theory. Seminal texts are that of Barr & Wells [3] for applications in computer science and Mac Lane's work in pure mathematics [21]. It is to be noted that their treatment is syntactical rather than semantic and the deep applied significance may not be too obvious in these syntactical descriptions.

Appendix I(a): Treatment by Barr and Wells

The classical approach as followed by Barr & Wells ([3] pp.142-160) defines a category **C** as Cartesian Closed if it satisfies the three conditions reproduced from their description in Panel 2.

Panel 2 : Three Conditions for a Cartesian Closed Category ([3] p.143)

CCC-1	There is a terminal object 1
CCC-2	Each pair of objects A and B of **C** has a product $A \times B$ with projections $p_1 : A \times B \to A$ and $p_2 : A \times B \to B$
CCC-3	For every pair of objects A and B, there is an object $[A \to B]$ and an arrow $eval : [A \to B] \times A \to B$ with the property that for any arrow $f : C \times A \to B$, there is a unique arrow $\lambda f : C \to [A \to B]$ such that the composite $C \times A \xrightarrow{\lambda f \times A} [A \to B] \times A \xrightarrow{eval} B$ is f

Traditionally the family of arrows (historically known as a Hom functor) from A to B is written as $[A \to B]$ or denoted as B^A and then called the exponential object with A as the exponent. It is possible to add some semantic detail to the statements CCC-1 to CCC-3 in the panel and draw formal diagrams to indicate further aspects. In basic terms the definition above requires a terminal object T as an upper limit closing the category from above. This has to be independently defined for the category of sets because there is no syntactical connection between the extension and the intension of a set. It lies in the semantics unexpressed and the connection has to be made in the mind of the user. A natural category on the other hand exists as an intension identity arrow typing, by means of a contravariant arrow, every object in its possible extensions. A pair of objects has a product with projections where there is only one path between the product and the related object. More precisely:

CCC-1 For any object A in the category, there is exactly one arrow $A \to T$, where T is the terminal object and the category is closed on top T. This is quite straight forward in finitary categories where the elements of a set are defined as independent of one another and can only be related by functions. In natural categories there is no such independence because of the nature of process every object in the world is related to every other. The semantics of CCC1 would then express the wholeness of the category.

CCC-2 expresses the property that any pair of objects may combine and any such combination may be resolved into one or other of its components. This appears fairly obvious at the syntactical level but provides the basis of relationships at the semantic level. Any combination is dependent on context which qualifies any relationship.

The first limb of **CCC-3** provides for currying to change a function on two variables to a function on one variable. For function $f : C \times A \to B$, let $[A \to B]$ be the set of functions from A to B. Then there is a function: $\lambda f : C \to [A \to B]$ where $\lambda f(c)$ is the function whose value at an element $a \in A$ is $f(c; a)$. This is equivalent to the typed lambda calculus. Typical examples of currying with integers often given are:

f : multiply(_, 2) $\to R$ curries to λf : double(_) $\to R$

f : exponentiate(_, 2) $\to R$ curries to λf : square(_) $\to R$

The use of ‘double’ and ‘square’ are examples of semantic expressions used to bridge conceptually the gap between intension and extension in set theory. This is the finitary syntactical version of the property in the universe that there is a single direct connection between any pair of two entities that is the resultant of all possible connections between them as illustrated in the diagram of Figure 3. The language used by Barr & Wells in these definitions is not purely categoreal but as not uncommon in finitary category theory it is often necessary to resort to hybrid descriptions involving set theoretic concepts as with the use here of lambda calculus, invented by Church to express for the purposes of set theory the concept of typing as a limit. Lambda calculus was known, from early on and for similar reasons, to be logically inconsistent. It is subject to the Kleene-Rosser paradox, which is another incarnation of Russell's paradox.

In the second limb of **CCC-3**, for every pair of objects A and **B**, there is an object $[A \to B]$ and an arrow eval : $[A \to B] \times A \to B$ with the property that for any arrow $f : C \times A \to B$ there is a unique arrow $\lambda f : C \to [A \to B]$ such that the diagram in Figure 15 commutes.

In Figure 15 C is the product object and eval is a function mapping all A objects and their associated B objects onto B. The semantics is very profound in that it leads to the Heyting logic mentioned previously which is only possible in finitary category theory by arbitrary enhancement but is naturally inherent in process categories where it is essentially the metaphysics of causation.

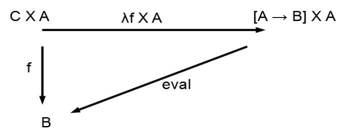

Fig. 15: Commuting Diagram for Rule CCC-3 (second limb) for a Cartesian closed category

The problems which arise from the lack of formal integrity between the intension of a set and the extension of its elements carry over into the concept of ‘locally Cartesian closed’. Natural categories have the property of being both Cartesian closed and locally Cartesian closed. As arbitrary models finitary categories may have the former property without the latter. Categories with both properties are treated as strong and those that are not also locally Cartesian closed as weak. In the former products are extended to pullbacks and Barr & Wells rely on this to distinction to define locally Cartesian closed ([3] at p.353). Categories are locally Cartesian closed when the category **C** has pullbacks and either the pullback functor has a right adjoint or for every object A in **C**, the slice category **C**/**A** is Cartesian closed. Pullbacks express relationships over objects in a particular context so locally Cartesian closed categories provide more expressiveness for finitary categories in representing the real world. Figure 16 compares the product and pullback.

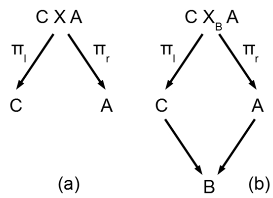

Fig. 16: Comparison of Constructions (a) Product $C \times A$ and (b) Pullback $C \times A$ in context of B

Some greater insight on their application to the real world comes from the first chapter in volume I of Peter Johnstone's *Sketches of an Elephant* [16]. A category is Cartesian closed if it has a terminal object, products of pairs of objects and equalizers of pairs of morphisms. A category is locally Cartesian closed if it has a terminal object and pullbacks of pairs of morphisms ([16] A1.2 p.11). A Cartesian closed category is locally Cartesian closed if it has pullbacks. The property of Cartesian-ness is stable under slicing ([16] A1.2.6). That is the stability functor Δ is in adjointness with the existential functor $\exists \dashv \Delta$ and with the universal functor $\Delta \dashv \forall$ for a pullback category. The approach by Barr & Wells to Cartesian closed categories can be adjusted to a more abstract view using adjointness. In the potentially adjoint relationship $F \dashv G$, the free functor F creates binary products and the underlying functor G checks for exponentials, that is one path. The free functor $\_\times A$ takes an object C to its product with A, that is $C \times A$. The underlying functor G takes a product object $C \times A$ to an object B. Figure 17 shows the diagrams that must both commute for adjointness to hold, diagram (a) for the left adjoint and (b) for the right adjoint. A comparison of Figures 15 and 17(b) shows that in the former the arrows $\_\times A(f)$ and ε correspond respectively to $\lambda f \times A$ and eval in the former. The counit of the adjointness is therefore the evaluation map.

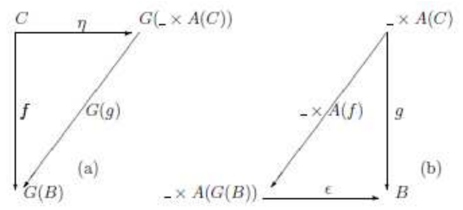

Fig. 17: Roles in Adjointness of a) η, the unit and b) ε, the counit of adjointness respectively. Free functor is $\_ \times A$.

Finitary Cartesian closed categories can be readily extended from binary products to finite products and this is demonstrated by Barr & Wells ([3] pp.191-196). For any objects A_1, A_2, ... , and A of a Cartesian closed Cartesian closed category, there is an object $[A_1 \to A]$ and an arrow:

$$eval : [A_1 \to A] \times A_1 \to A$$

such that for any $f : \pi_{A_2} \to A$ there is a unique arrow:

$$\lambda_1 f : \pi_{A_2} \to A[A_1 \to A]$$

Finite products give construction of n-tuples which Barr &Wells [3] show can represent strings through constructions such as the Kleene closure (p.340) and the Kleisli category (pp.366-367). These seem attractive but being derived from sets they have to be treated with caution for use in information systems. For they are still models and prone to the same difficulties we have already discussed. For instance the issue arises with the significance of order. This is not new. Historically there has been some debate about whether $A\times B$ is 'the same' as $B\times A$. Barr & Wells for instance are compelled to acknowledge the difficulty with finite products as in Panel 3.

Panel 3 : Problems with Equivalence of Products in CCC-3 ([3] p.144)

Condition CCC-3 appears to treat the two factors of $C\times A$ asymmetrically, which is misleading since of course $C\times A \equiv A\times C$. Even that last isomorphism is misleading since $C\times A$ and $A\times C$ could be taken to be the same object. Products are of indexed sets of objects, not necessarily indexed by an ordered set, even though our notation appears to suggest otherwise. It gets even worse with *n*-ary products ...

In applications such as relational databases a product is regarded as an associative operation so that $A \times (B \times C)$ is regarded as equivalent to $(A \times B) \times C$, at least at the data level. But this is the problem: extensionally the product operation is associative. However, intensionally a different answer is obtained depending on the order of the operations. So the product operation is not associative in Cartesian closed systems.

Appendix I(b): Treatment by Mac Lane

Mac Lane ([21], pp.87-88) defines Cartesian closed in tabular form using the diagonal functor Δ for product and the terminal object in category C in **Set** as reproduced here in Panel 4.

Mac Lane asserts the existence of a Cartesian Closed Category as equivalence with adjointness, as in Panel 5.

Panel 4 : Left and Right Adjoints in Cartesian Closed Category C in Set after ([21] pp.87-88)

Functor	*Adjoint*	*Unit*	*Counit*
$\Delta : C \longrightarrow C \times C$	Left: Coproduct $\amalg : C \times C \longrightarrow C$ $< a, b > \longmapsto a \amalg b$ Right: Product $\Pi : C \times C \longrightarrow C$ $< a, b > \longmapsto a \times b$	(pair of) injections $i : a \longrightarrow a \amalg b$ $j : b \longrightarrow a \amalg b$ Diagonal arrow $\delta_c : c \longrightarrow c \times c$ $x \longmapsto < x, x >$	'folding map' $c \amalg c \longrightarrow c$ $ix \longmapsto x, jx \longmapsto x$ (pair of) projections $p : a \times b \longmapsto a$ $q : a \times b \longmapsto b$
$C \longrightarrow \mathbf{1}$	Left: Initial object s Right: Terminal object t	 $c \longrightarrow t$	$s \longrightarrow c$

Panel 5 : Assertion of Cartesian Closed Category as Equivalence with Adjointness ([21] p.97)

To assert that a category C has all finite products and coproducts is to assert that products, terminal, initial and coproducts exist, thus the functors $C \to 1$ and $\Delta : C \to C \times C$ have both left and right adjoints. Indeed the left adjoints give initial object and coproduct, respectively, while the right adjoints give terminal object and product, respectively.

Mac Lane ([21] at pp.97-98) thus by using just adjoints at both the category level and the object level is able to define "Cartesian Closed Category'. He puts it this way: a category **C** with all finite products specifically given is called *Cartesian closed* when each of the following functors in Panel 6 has a *specified* right adjoint (with a specified adjunction) in Panel 7.

Panel 6 : Functors and Maps involved in Adjointess ([21] p.98)

$C \longrightarrow \mathbf{1}$,	$C \longrightarrow C \times C$,	$C \xrightarrow{-\times b} C$,
$c \longmapsto 0$,	$c \longmapsto < c, c >$,	$a \longmapsto a \times b$,

The first adjoint in Panel 7 specifies the terminal object and the second the product and its projections. The third specifies the evaluation map as shown in Panel 8.

Panel 7: Right Adjoints for Cartesian Closed Category ([21] p.98)

Panel 8 : Evaluation map as condition for adjointness in Cartesian Closed Category ([21]p.98)

The third required adjoint specifies for each functor $- \times b : C \longrightarrow C$ a right adjoint, with the corresponding bijection
$\hom(a \times b, c) \equiv \hom(a, c^b)$
natural in a and in c. By the parameter theorem (to be proved in the next section), $< b, c > \longmapsto c^b$ is then (the object function of) a bifunctor $C^{op} \times C \longrightarrow C$. Specifying the adjunction amounts to specifying for each c and b an arrow e
$e : c^b \times b \longrightarrow c$
which is natural in c and universal from $- \times b$ to c. We call this $e = e_{b,c}$ the *evaluation* map.

Mac Lane's treatment, in common with that of Barr & Well's, is restricted to the category of **Set**. From the point of real-world systems such as information and database systems, this is unsatisfactory as in the Boolean world there is a reliance for negation on the closed world assumption. What is required is an open system, through the free functor *F*, with Heyting intuitionistic logic to give negation in an approach which does not violate Gödel's principles.

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