

LOGIC AND EMOTION: WHITEHEAD'S CATEGORY OF THE ULTIMATE

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ABSTRACT

Category theory has its foundations in the pure mathematics of sets. The philosopher Alfred North Whitehead developed an alternative basis in his book *Process and Reality* in the description of the Category of the Ultimate, with prehension employed for the *becoming* of *beings*. This paper argues that this category corresponds closely to a topos, prehension and *becoming* to adjointness and *being* to existence, all mainstream concepts in category theory. Whitehead's work therefore deserves to be treated as a philosophical basis for category theory with its additional emphasis on processes in general, including biological ones, offering a lead into the more adventurous use of category theory in the future. ¹

1 The Text

An *Essay in Cosmology* is the subtitle given by Alfred North Whitehead to his celebrated Gifford lectures for the title *Process and Reality* (P&R) at Edinburgh in the session of 1927-28. The work was formally published in 1929 by publishers in both Cambridge, UK, and New York [29]. There were some immediate issues. There were two conflicting original editions, with poor proof reading by Whitehead, his poor handwriting, and inconsistencies, all contributing to the problems. These issues were at least partially resolved in a corrected version, produced in 1978 and available on the Internet [30]. In this paper we use the edited version of 1978 as the text for consideration.

¹Michael Heather sadly passed away on 20 September 2022. He had already made a significant contribution to this work.

Whitehead develops his cosmology in terms of a Categorical Scheme, based on the philosophy of organism, described as speculative in nature. The foundation of his whole scheme of cosmology is the Category of the Ultimate. A category in process terms is a typing and this fundamental category of his “expresses the general principle presupposed in the three more special categories”.

An obvious question is: what does Whitehead mean by a category? This will be explored later but it appears to be basically an Aristotelian concept. He is certainly not using the term as found in category theory as it was not until 15 years later, in 1945, that that theory began to emerge in modern terms with the paper by Eilenberg and MacLane [3]:

In order to deal in a general way with such situations, we introduce the concept of a category. Thus a category will consist of abstract elements of two types: the objects A (for example, vector spaces, groups) and the mappings a (for example, linear transformations, homomorphisms). For some pairs of mappings in the category there is defined a product (in the examples, the product is the usual composite of two transformations). Certain of these mappings act as identities with respect to this product, and there is a one-to-one correspondence between the objects of the category and these identities. A category is subject to certain simple axioms, so formulated as to include all examples of the character described above. [p.234]

Figure 1(a) shows a category CAT with a terminal object, which has a unique arrow on to it from every object in the category. The dual of this category CAT^{OP} , in Figure 1(b), shows a typing arrow from the terminal object to every object in the category. This is the simplest example of typing as contravariant. For the terminal object is the identity functor, mapping from a category to itself, defining the category. It is to be noted that the arrow in category theory can be usefully interpreted as typing. The objects of Figure 1 could just as well themselves be categories in some higher category or even more powerfully as functors, between categories, in a yet higher category. The typing arrow may be a natural transformation, a mapping between functors, crossing more than one level.

P&R has been analysed by Alex Scott [23] from a philosophical perspective and we later make use of some of his interpretations,

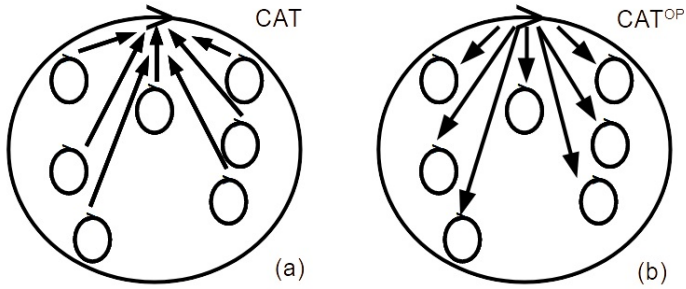


Figure 1. (a) A Category CAT with a Terminal Object; (b) The Dual Category CAT^{OP} with Typing Arrows

based also on the 1978 version. Whitehead extends his philosophical substance from physics to biology and it is relevant to mention the work of Robert Rosen, who in his book *Life Itself* explored categorical relations in biological systems [22]. In earlier work we interpreted Rosen's work using the diagram in Figure 2 showing transformations in biological systems [20]. We return to Rosen's work in the Discussion.

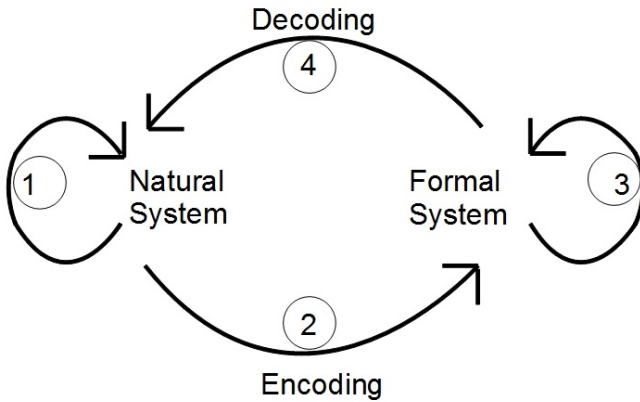


Figure 2. Modelling a Natural System with a Free System, adapted from Rosen [22] Arrows: ① causality, ③ implication, ②, ④ as labelled

The authors have visited this area over the past twenty years noting the broad similarity between Whitehead's categories and the categories of category theory. For instance in [5] at Salzburg in 2006, in [8] at Vilnius in 2013 and in [7] at ANPA in 2011 we looked

at topos theory from the angle of real-world requirements as opposed to pure mathematics, motivated by Whitehead's categories. At ANPA in 2017 we looked at Whitehead's *Process and Reality*, in particular Part IV, from a time-space perspective [9]. The paper presented to the Whitehead conference in Bangalore in 2009 [6] is a precursor to the current paper, examining Whitehead's Category classification in some detail but refraining from examining Whitehead's Categorical Scheme on a line by line basis.

2 Rationale

There are a number of reasons why we have delayed doing this very detailed analysis. Whitehead's writing style is dense, involving many of his own terms. He is also quite a prolific writer, so large bodies of text need to be examined. The virtual absence of diagrams and the almost complete absence of mathematics in his texts means that apparent ambiguities have to be resolved from linguistics alone.

However we feel that a closer look is warranted on his actual text. Our increased understanding of category theory may give an insight into some areas. We need to focus on concepts Whitehead considered important. Above all there is a desire to extend formalisms into biological sciences. Through rejecting the dualism of Descartes Whitehead unifies the physical and biological in philosophical terms.

Whitehead constructs a Categorical Scheme [P&R p.20] with special categories:

- Category of the Ultimate
- Categories of Existence
- Categories of Explanation
- Categorical Obligations

These special categories are composed of

- Eight categories of existence
- Twenty seven categories of explanation
- Nine categorical obligations

There is only one Category of the Ultimate: “The Category of the Ultimate expresses the general principle presupposed in the three more special categories” [P&R p.21]. The whole of P&R rests on this categorial scheme. At first sight there seems to be a hierarchical typing relationship among these categories which might look like this, as shown in Figure 3. The numbers in this diagram are the counts of categories of each type as given above.

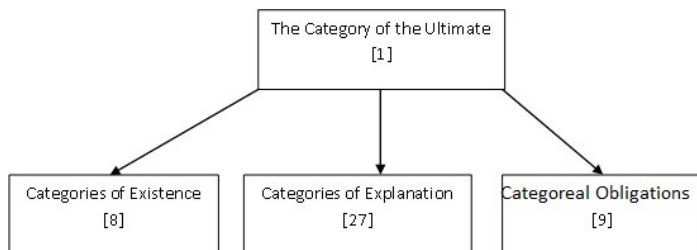


Figure 3. The Categorial Scheme of Whitehead: Top Level

There are problems with this hierarchy. From the formal theory of categories we can understand the need for interdependence between categories, which is not achieved in a hierarchy. So Whitehead presumably dismissed the use of hierarchies as natural relationships are more complex than the hierarchy. In the same way Russell used the phrase ‘ramified type-theory’ rather than ‘hierarchical type-theory’ although both words contain the sense of a tree.

It is equally important to look at what Whitehead does not say. He does not provide such a diagram in P&R nor indeed any diagram in P&R until much later in the text where a few are produced, described as ‘Extensive Connection for diagrams’ [P&R pp.295-296]. Nor does he state that there is a hierarchical relationship between the main categories in the Scheme. Whitehead does not even explain what he means by the term ‘category’. It seems it is defined by the Category of the Ultimate itself and therefore is self-referencing. However at [P&R p.21] there is a strong suggestion that the category may be Aristotelian: “This Category of the Ultimate replaces Aristotle’s category of ‘primary substance’ ”. We consider the primary substance to be instances corresponding to extension with the secondary substance being intension or classification [26]. It does not appear that a category is to be identified with a set. For instance there does not appear to be any difficulty with a category being a member of itself, which caused such

problems in his other magnum opus (co-authored with Bertrand Russell) the *Principia Mathematica* (PM) [28].

Whitehead seems very conscious of the speculative nature of his philosophy at this stage of its development. The whole of Part I of P&R is headed 'a Speculative Scheme'. It is speculative perhaps because at the time he was only just giving the Gifford lectures and for the remaining 20 years of his life there was no formal presentation available for P&R as he and Russell were able to provide for PM.

Whitehead developed a one-substance approach in contrast to Descartes [2], and many other philosophers, who considered there were two substances: material body and mental (mind, thought). Whitehead abandoned this approach, considering there was just one substance, so unifying mind and matter as a single process. Whitehead describes his system of speculative philosophy as a 'philosophy of organism' or 'organic philosophy,' since he views reality as consisting of interrelated and mutually dependent parts that are involved in sustaining vital processes [P&R p.19]:

With the purpose of obtaining a one-substance cosmology, 'prehensions' are a generalization from Descartes' mental 'cogitations,' and from Locke's 'ideas' to express the most concrete mode of analysis applicable to every grade of individual actuality. Descartes and Locke maintained a two-substance ontology - Descartes explicitly, Locke by implication. Descartes, the mathematical physicist, emphasized his account of corporeal substance; and Locke, the physician and the sociologist, confined himself to an account of mental substance.

It will be noted that prehension is a core building block in the Categorical Scheme, which is perhaps no longer a speculative philosophy as new formal techniques are available to underpin it: the formal theory of categories of Eilenberg and MacLane [3], only just beginning at the time of Whitehead's death in 1947 and maybe satisfying but now a maturing category theory foreshadowed by him in his *Category of the Ultimate* quite comprehensively in the sense of his preface:

Motivation for a complete cosmology - to construct a system of ideas which bring the aesthetic, moral and religious interest into relation with those concepts of the world which have their origin in natural science.
[P&R p.vi]

3 The Eight Categories of Existence

The Eight Categories of Existence are given by Whitehead [P&R p.22] as:

- (i) Actual Entities (also termed Actual Occasions), or Final Realities, or Res Verae
- (ii) Prehensions, or Concrete Facts of Relatedness
- (iii) Nexūs (plural of Nexus), or Public Matters of Fact
- (iv) Subjective Forms, or Private Matters of Fact
- (v) Eternal Objects
- (vi) Propositions, or Matters of Fact in Potential Determination, or Impure Potentials for the Specific Determination of Matters of Fact, or Theories.
- (vii) Multiplicities, or Pure Disjunctions of Diverse Entities
- (viii) Contrasts, or Modes of Synthesis of Entities in one Prehension, or Patterned Entities

Among these eight categories of existence, actual entities and eternal objects stand out with a certain extreme finality. The other types of existence have a certain intermediate character. The eighth category includes an indefinite progression of categories, as we proceed from 'contrasts' to 'contrasts of contrasts' and on indefinitely to higher grades of contrasts.

We look in more detail at each of these categories in turn. The notation preserves quote marks faithfully in Whitehead's quotes but in our comments Whitehead's terms such as *being*, *becoming* and *feeling*, which are frequent in normal language are italicised to reduce the number of quote marks.

3.1 Subjective Forms: Entities, Prehension and Concrescence

Since subjective forms involve entities and prehension, and concrescence is a fundamental property of prehension, we first consider Whitehead's Categories of Existence (i), (ii) and (iv) together. Whitehead commences his exploration in detail with a consideration of entities at [P&R p.20]:

Actual entities involve each other by reason of their prehensions of each other. There are thus real individual facts of the togetherness of actual entities, which are real, individual, and particular, in the same sense ... which actual entities and the prehensions are real, individual, and particular. Any such particular fact of togetherness among actual entities is called a 'nexus' (plural form is written 'nexūs'). The ultimate facts of immediate actual experience are actual entities, prehensions, and nexus. All else is, for our experience, derivative abstraction.

We observe that entities are real with existence, individual as atoms, particular as can be singled out with identity. Entities can be joined together as a nexus, which appears to be a union of connected types such as an ordered society. Every entity should be a specific instance of one category of existence, giving a basic classification or typing. At [P& R p.22] some synonymy is developed in the Category of Existence: actual Entities (also termed Actual Occasions), or Final Realities, or Res Veræ (true thing). Category of Explanation (i) states: "That the actual world is a process, and that the process is the becoming of actual entities. Thus actual entities are creatures; they are also termed 'actual occasions' ". *Becoming* is the creative process and occasions are now (in time).

An important question is: what is prehension? This is the second Category of Existence: "Prehensions, or Concrete Facts of Relatedness". It is an overloaded word, meaning grasping or seizing, as in prehensile tails of some monkeys, or an interaction of a subject with an event or entity which involves perception but not necessarily cognition. Whitehead provides more detail at [P&R p.19]:

A prehension reproduces in itself the general characteristics of an actual entity: it is referent to an external world, and in this sense will be said to have a 'vector character'; it involves emotion, and purpose, and valuation, and causation. In fact, any characteristic of an actual entity is reproduced in a prehension. It might have been a complete actuality; but, by reason of a certain incomplete partiality, a prehension is only a subordinate element in an actual entity.

and defines it more concretely at [P&R p.23]:

- (xi) That every prehension consists of three factors: (a) the 'subject' which is prehending, namely, the actual entity in which that prehension is a concrete element; (b) the 'datum' which is prehended; (c) the 'subjective form' which is how that subject prehends that datum.

In formal category theory terms there is little doubt that Whitehead is describing here a pullback or a pushout, representing the interconnectedness of the world. A pullback represents a product relationship between the independent variable (Whitehead's subject (a), say A) and a dependent variable, say D , over a data object (Whitehead's datum (b), say B), to give a restricted product (Whitehead's subjective (c)) of the pullback C as $A \times_B D$. In a pushout the \times is replaced by $+$, so C would be $A +_B D$. So the subjective form C prehends B into A to give the new form $A \times_B D$ or $A +_B D$. In the category of sets the former is a limit cone, based on a subset of the Cartesian product $A \times D$, and the latter a colimit cone, based on a subset of the disjoint union $A + D$. A prehension at this stage of our analysis is therefore a restricted product as in Figure 4 or a restricted coproduct as in Figure 5.

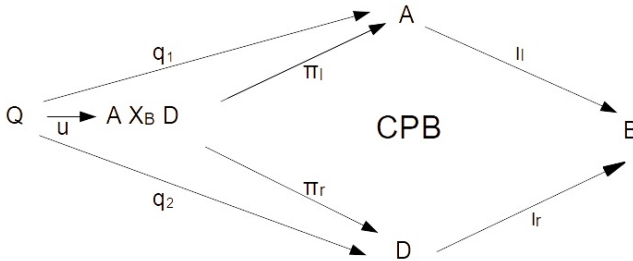


Figure 4. Category CPB : the Limit Diagram of Category C : the Pullback $A \times_B D$

The diagram Figure 4 for the pullback category CPB illustrates a number of features of category theory: the diagram commutes through different paths between objects yielding an equivalent result, that is $\iota_l \circ \pi_l = \iota_r \circ \pi_r$; $q_1 = \pi_l \circ u$ and $q_2 = \pi_r \circ u$. The restricted product $A \times_B D$ is projected by π into its component category-objects, to the left through π_l as A and to the right through π_r as D ; the category-objects A and D are included by ι in the coproduct, written conventionally in shorthand as B but in truth being the coproduct $A + D + B$; ι_l is the left inclusion and ι_r is the right

inclusion. The lower-bound, the limit, ensures the diagram is natural (universal) with a unique morphism u from the limit to the restricted product $A \times_B D$, providing a factorisation through the whole diagram.

The diagram in Figure 5 for the pushout category CPO has the equations $\iota_l \circ f = \iota_r \circ g$; $q_1 = u \circ \iota_l$ and $q_2 = u \circ \iota_r$. The restricted coproduct $A +_B D$ is the amalgamated sum of A and D produced by the injections ι_l and ι_r in the context of B . The upper-bound, the colimit, ensures the diagram is natural (universal) with a unique morphism u to the colimit from the restricted coproduct $A +_B D$, providing a factorisation through the whole diagram. The objects in both diagram may be categories, with their own internal structure of arrows, termed category-objects.

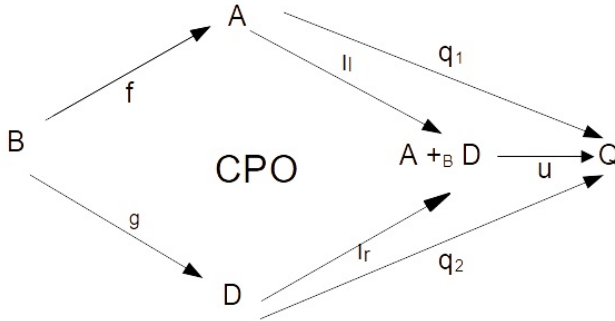


Figure 5. Category CPO : the Colimit Diagram of Category C : the pushout $A +_B D$

In general Whitehead describes prehensions as concrete modes of analysis of the world: to prehend something is to have a concrete idea or concept of that thing. However, prehension is not merely a mode of thinking: a prehension is a process of appropriation of an element of an actual entity or of an element that is derived from an actual entity. A prehension of an object or of an element of an object changes the internal constitution of the prehending subject. Prehension is a process by which an actual entity, or prehending subject, becomes itself by appropriating elements from other actual entities. The *becoming* of an actual entity occurs through a concrecence of prehensions. Satisfaction is a final phase of concrecence (or the process of integration of *feeling*), in which prehensions are integrated into a concrete unity. A *feeling* is the integration of an actual entity or occasion into the internal constitution of a subject.

This reinforces our earlier understanding with the addition of the term concrescence which appears to be the closing adjointness in an enhanced Cartesian closed category: the locally Cartesian closed category (LCCC). We have adjointness between the free functor \exists creating a new emergent entity and the underlying pullback functor Δ performing integrity checks as shown in Figure 6. So prehension is handled by a pullback (or pushout) with an additional adjoint relationship to construct a locally Cartesian closed category. Note also Whitehead's use of the term *feeling*, which introduces biology into his structures.

For Whitehead, prehension is how all *feelings* are felt, or if you prefer, how all data is transferred. It is ontologically primary, in that all process is prehension of one actuality by another actuality. Causal efficacy is a species of prehension. Sense perception is a species of prehension. The vast majority of prehensions never makes it to consciousness, which only occurs in the final phase of an event's concrescence. So, prehensions may have a causal effect, but not consciously. All Prehensions are *beings* that are potential *becomings*: it is concrescence that converts a virtual *being* into a real one, a *becoming*.

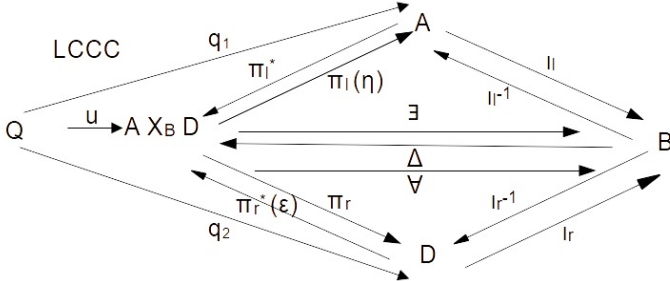


Figure 6. The locally Cartesian closed category *LCCC*: Adjointness in the functors between the product $A \times_B D$ and B

The diagram in Figure 4 can be converted into the LCCC of Figure 6 (labelled *LCCC*) by showing the functors, between the restricted product $A \times_B D$ and the coproduct $B + A + D$, as the hyperdoctrine of Lawvere [12] $\exists \dashv \Delta \dashv \forall$ with adjointness between the functors: the existential quantifier \exists is left adjoint to the diagonal Δ , which is in turn left adjoint to the universal quantifier \forall . The universal quantifier is right adjoint to the diagonal Δ , which is in turn right adjoint to the existential quantifier \exists . The quantifiers \exists and \forall provide the search facilities of an information

retrieval or database system. The diagonal functor Δ identifies pairs in the relationship between \mathbf{A} and \mathbf{D} in the context of \mathbf{B} . The opposite arrows are shown with inverse projections as π^* and inverse inclusions as ι^{-1} . Two properties, part of the adjointness in a 4-tuple relation, for example $\langle \exists, \Delta, \eta, \epsilon \rangle$, remain to be defined: η is the unit of adjunction measuring freeness through π_l in the diagram and ϵ is the counit of adjunction measuring co-freeness through π_r^* . The category-objects in a LCCC are typically further LCCC so a category-object at the top-level can be decomposed into further category-objects, each representing more detail in the application. A locally Cartesian closed category therefore provides a recursive feature. Figure 4 shows a *being*, a tentative *becoming*; Figure 6 shows a *becoming*: the adjointness giving final satisfaction in the concrecence. Figure 6 is the subjective form showing how the datum in the category B is prehended by the category A .

Whitehead later provides further information which indicates that he had the modern concept of adjointness in mind:

There is a prevalent misconception that ‘becoming’ involves the notion of a unique seriality for its advance into novelty. This is the classic notion of ‘time’ which philosophy took over from common sense. [P&R p.35]

The ‘prehension’ of one actual entity by another actual entity is the complete transaction, analysable into the objectification of the former entity as one of the data for the latter, and into the fully clothed feeling whereby the datum is absorbed into the subjective satisfaction – ‘clothed’ with the various elements of its ‘subjective’ form. [P&R p.52].

The first part is consistent with our understanding of LCCC: adjointness is not serial, but is a snap. The second part shows how adjointness provides satisfaction through a hyperdoctrine as the final stage in *becoming*. Furthermore, some data are negatively prehended, which according to Whitehead’s principle of relativity, must have some influence on the concrecing occasion, but according to his description, must be vanishingly small, as neither the prehensions content nor its form are integrated into concrecence.

In his *Categories of Explanation*, Whitehead emphasises these two types of prehension, positive and negative:

- (xii) That there are two species of prehensions: (a) ‘positive prehensions’ which are termed ‘feelings,’ and (b)

‘negative prehensions’ which are said to ‘eliminate from feeling.’ [P&R pp.23-24].

(xxvii) In a process of condescence, there is a succession of phases in which new prehensions arise by integration of prehensions in antecedent phases. In these integrations ‘feelings’ contribute their ‘subjective forms’ and their ‘data’ to the formation of novel integral prehensions; but ‘negative prehensions’ contribute only their ‘subjective forms’. The process continues till all prehensions are components in the one determinate integral satisfaction. [P&R p.26].

From a categorial perspective positive prehensions involve the whole of a pullback diagram such as in Figure 6 above so have limits and commute, in accordance with the rules of category theory. Negative prehensions represent only the relationship $\mathbf{A} \times_{\mathbf{B}} \mathbf{D}$ without the data B and appear rather set-like as they do not participate in commuting diagrams. The authors have not incorporated negative prehensions into the category theory representing Whitehead’s ideas; the idea that negative prehensions are without feeling and hence inoperative, strikes some resonance with category theory.

3.2 A Nexus of Entities with Defining Characteristic

Nexus is category (iii) in the Categories of Existence. Above it can be seen that “Any such particular fact of togetherness among actual entities is called a ‘nexus’ ” [P&R p.20 (classification)] and concluded that a nexus appears to be a union of connected types such as an ordered society. Whitehead gives further information on nexus at [P&R p.24]:

(xiv) That a nexus is a set of actual entities in the unity of the relatedness constituted by their prehensions of each other, or — what is the same thing conversely expressed - constituted by their objectifications in each other.

(xv) That a proposition is the unity of certain actual entities in their potentiality for forming a nexus, with its potential relatedness partially defined by certain eternal objects which have the unity of one complex eternal object. The actual entities involved are termed the ‘logical subjects’, the complex eternal object is the ‘predicate’.

A nexus is described as a set of actual entities. In modern terms a set may be viewed more profitably as a collection or a category. The relationship is through a prehension, implying the entities share a limit (through a restricted product) or a colimit (through a directed sum) in a Cartesian closed category. This category is not necessarily locally Cartesian closed as concrescence does not appear to be a required feature. The unity of one complex eternal object suggests that we are indeed dealing with a limit or a colimit. Whitehead continues:

(iv) The Category of Conceptual Valuation. From each physical feeling there is the derivation of a purely conceptual feeling whose datum is the eternal object determinant of the definiteness of the actual entity, or of the nexus, physically felt. [P&R p.26]

Here Whitehead is, within his single substance approach, showing how physical feelings give rise to conceptual ones. Whitehead provides more detail at [P&R p.34], likening a nexus to a society with order:

The notions of ‘social order’ and of ‘personal order’ cannot be omitted from this preliminary sketch. A ‘society’ in the sense in which that term is here used, is a nexus with social order; and an ‘enduring object’ or ‘enduring creature’ is a society whose social order has taken the special form of ‘personal order.’

A nexus enjoys ‘social order’ where (i) there is a common element of form illustrated in the definiteness of each of its included actual entities, and (ii) this common element of form arises in each member of the nexus by reason of the conditions imposed upon it by its prehensions of some other members of the nexus, and (iii) these prehensions impose that condition of reproduction by reason of their inclusion of positive feelings of that common form. Such a nexus is called a ‘society’ and the common form is the ‘defining characteristic’ of the society. The notion of ‘defining characteristic’ is allied to the Aristotelian notion of ‘substantial form’.

A nexus enjoys ‘personal order’ when (a) it is a ‘society’ and (b) when the genetic relatedness of its members orders these members ‘serially’.

Thus the nexus forms a single line of inheritance of its defining characteristic.

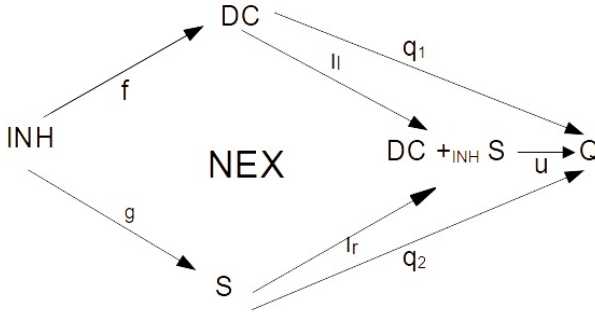


Figure 7. The Colimit Diagram of Category NEX : the Pushout $DC +_{INH} S$

Since sets are not inherently ordered, Whitehead is clearly not looking for a set-based solution. Categories have an inherent ordering, more naturally suited for representing a social order. The common form for a society, its defining characteristic, is likened to Aristotle's substantial form. Such a form is an Aristotelian second substance, in our terms the intension or class of a collection of values. So a nexus NEX is a pushout category $DC +_{INH} S$ whose contents are defined by an intension or type-category [26] with defining characteristic DC , a society category S and a linking inheritance category INH creating the directed sum for pairs of DC and S , as in Figure 7 where $\iota_l \circ f = \iota_r \circ g$; $q_1 = u \circ \iota_l$ and $q_2 = u \circ \iota_r$. It is apparent that we are dealing with union here rather than product. The authors use NEX through prehensions as directed sums in B in Figure 6, the whole being governed by a colimit. This meets the general understanding of nexus as provided by dictionaries and other sources, for example:

- in the Britannica Dictionary: “definition of NEXUS [singular] formal : a relationship or connection between people or things — often + between or of [e.g.] the nexus between teachers and students ...”.
- in the Merriam-Webster dictionary: “Definition of nexus 1: CONNECTION, LINK the nexus between teachers and students also : a causal link, the nexus between poverty and crime ...”

- in the Oxford dictionary: “a complicated series of connections between different things the nexus between industry and political power ...; from Latin, ‘a binding together’, ...”
- in work by Pries-Heje and Baskerville [19]: “Managers frequently face ill-structured or “wicked” problems. Such problems are characterized by a large degree of uncertainty with respect to how the problem should be approached and how to establish and evaluate the set of alternative solutions. A design theory nexus is a set of constructs and methods that enable the construction of models that connect numerous design theories with alternative solutions. It thereby offers a unique problem-solving approach that is particularly useful for addressing ill-structured or wicked problems.”

3.3 Multiplicities as Heterogeneous Collections

“Multiplicities, or Pure Disjunctions of Diverse Entities”, are category (vii) in the Category of Existence. Further explanation of this term is presented at [P&R p.24]:

(xvi) That a multiplicity consists of many entities, and its unity is constituted by the fact that all its constituent entities severally satisfy at least one condition which no other entity satisfies. Every statement about a particular multiplicity can be expressed as a statement referent either (a) to all its members severally, or (b) to an indefinite *some of* its members severally, or (c) as a denial of one of these statements. Any statement, incapable of being expressed in this form, is not a statement about a multiplicity, though it may be a statement about an entity closely allied to some multiplicity, i.e., systematically allied to each member of some multiplicity.

The phrase ‘at least one condition’ indicates that multiple conditions occur but that at the lower extreme of one, the condition becomes closer to the defining characteristic of a nexus. Rule (a) expresses the standard statement that a condition applies to all members severally, rule (b) suggests that this rule may be relaxed in unknown circumstances, rule (c) indicates that the statement may operate with negative logic.

From a philosophical point of view, Whitehead's 'philosophy of organism' may be described as a metaphysical pluralism, insofar as he views reality as consisting of a multiplicity of actual entities [23]. Multiplicities are similar to nexus but in the latter we have a single defining characteristic for providing a type to the members of a society as in Figure 7. A multiplicity therefore contains heterogeneous entities with no defining type characteristic. Whitehead uses the term single inheritance for the defining characteristic, and multiple inheritance for a multiplicity. However, he also implies that a multiplicity reduces to a nexus if there is only one condition.

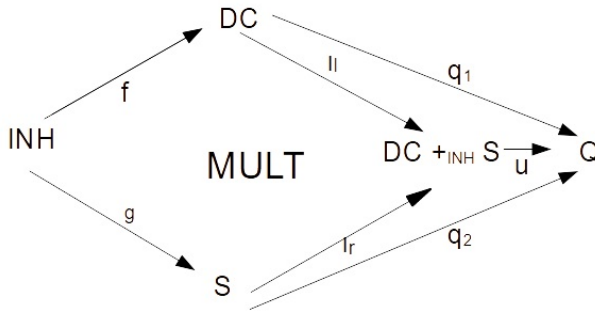


Figure 8. The Colimit Diagram of Category *MULT*: the Pushout $DC +_{INH} S$

The category for multiple inheritance *MULT* shown in Figure 8 is a similar colimit to that for single inheritance *NEX* but the relationship *g* between *INH* and *S* changes from 1:1 (each inheritance is in 1 society, each society has 1 inheritance) to N:M (each inheritance can be in M societies, each society has N inheritances). The authors retain *DC* as a potential type definition with *f* continuing to be 1:1 (each inheritance has one type, each type has one inheritance).

3.4 Eternal Objects as Constant Functors

Eternal objects are category (v) in the Categories of Existence. Whitehead introduces eternal objects at [P&R pp.22-23]:

- (iii) That in the becoming of an actual entity, novel prehensions, nexus, subjective forms, propositions, multiplicities, and contrasts, also become; but there are no novel eternal objects.

(vii) That an eternal object can be described only in terms of its potentiality for ‘ingression’ into the becoming of actual entities; and that its analysis only discloses other eternal objects. It is a pure potential. The term ‘ingression’ refers to the particular mode in which the potentiality of an eternal object is realized in a particular actual entity, contributing to the definiteness of that actual entity.

(xi) ... prehensions whose data involve actual entities — are termed ‘physical prehensions’; and prehensions of eternal objects are termed ‘conceptual prehensions’. Consciousness is not necessarily involved in the subjective forms of either type of prehension.

At [P&R p.44] Whitehead indicates that the main defining characteristic of an eternal object is that it is atemporal:

Any entity whose conceptual recognition does not involve a necessary reference to any definite actual entities of the temporal world is called an ‘eternal object’

Eternal objects are therefore time-invariant, so are constant and moreover are fixed at the outset with no new creations permitted. Eternal objects may participate in prehensions but the result is a conceptual prehension with no actuality of their own, as opposed to the physical prehensions, resulting from actual entities. Cobb notes that eternal objects are nontemporal, they do not come into *being* and do not pass away [1]. Whitehead indicates that eternal objects may also be identified with a unity complex eternal object at [P&R p.24]:

(xv) That a proposition is the unity of certain actual entities in their potentiality for forming a nexus, with its potential relatedness partially defined by certain eternal objects which have the unity of one complex eternal object. The actual entities involved are termed the ‘logical subjects’ the complex eternal object is the ‘predicate’.

This means that eternal objects include sensory qualities, like colours and tactile sensations; conceptual abstractions like shapes; numbers; moral qualities; physical fundamentals; feelings like an emotion, aversion, pleasure or pain; qualia [24].

In category theory, the constant functor [16] maps from one category to another, for instance from a source category CAT into a target category E , holding the eternal objects. A constant functor $\Delta e : CAT \rightarrow E$ is a functor that maps each object of the category CAT to a fixed object $e \in E$ and each morphism of CAT to the identity morphism of that fixed object. The notation Δ , a diagonal functor [17], indicates that a product is involved in the categorical construction for external objects ETO as shown in Figure 9. Of course this categorical construction only captures the surface structure of an eternal object: it does not capture the semantics underlying each facet such as sensory objects or feelings itemised above. The authors do explore this area later.

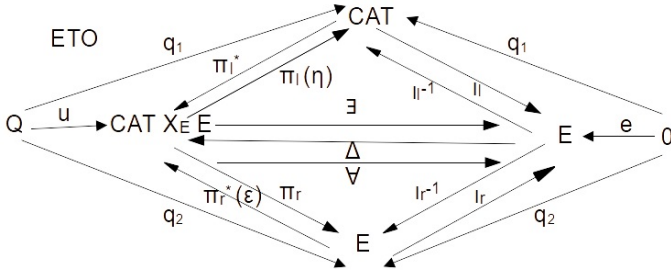


Figure 9. The Eternal Object Category as a locally Cartesian closed category ETO : Adjointness in the Functors between the Product $CAT \times E$ and the Category E

This diagram has a central line of direction: $0 \rightarrow E \rightarrow CAT \times E$. The limit is shown as there is trivially a right adjoint $\Delta \dashv \exists$. The 0 is a handle through identity on E , providing the unity of the one complex eternal object of Whitehead. The diagram is a topos, a structure discussed further later, with both a limit cone centred on Q and a colimit cone centred on 0 . The diagram ETO can be used for ingress into prehension by incorporating it as part of the structure of B in Figure 6.

3.5 Contrasts: Categories of Categories

In introducing the eight Categories of Existence above, Whitehead [P&R p.22] says “The eighth category includes an indefinite progression of categories, as we proceed from ‘contrasts’ to ‘contrasts of contrasts’ and on indefinitely to higher grades of contrasts”. Novel contrasts are also permitted as recorded above for

Eternal Objects. Contrast has a similar meaning to compare, a frequent description of functors and natural transformations in category theory.

In category theory a category of category construction is well-established as higher-order category theory with the notion of the n -category [13] where n is the level of addressing. So $n = 2$ gives a category of category construction, $n = 3$ gives a category of category of category construction, etc. Leinster describes a category with $n \geq 2$ as a multicategory, which behaves like a basic category except that the domain of an arrow is not just a single object but a finite sequence of them. In higher-dimensional category theory an n -category consists of 0-cells (objects) a, b, \dots ; 1-cells (arrows) f, g, \dots ; 2-cells (arrows between arrows) α, β, \dots , 3-cells (arrows between arrows between arrows) Γ, Δ, \dots , where $n = 3$. A basic category is a 1-cell, adjointness and natural transformations are 2-cells and a 3-cell involves a mapping between two natural transformations as shown in Figure 10.

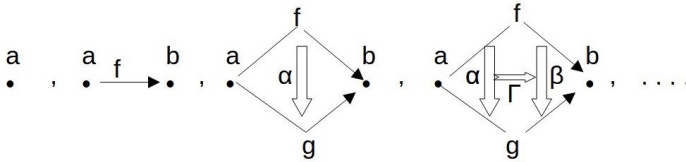


Figure 10. n -cells in Higher-order Category Theory, after Leinster [13] p.vi

The nesting of arrows in n -categories matches the nesting of contrasts in Whitehead's eighth Category of Existence.

3.6 Propositions

Category (vi) in the Categories of Existence is entitled 'Propositions, or Matters of Fact in Potential Determination, or Impure Potentials for the Specific Determination of Matters of Fact, or Theories'. Propositions appear to lead into the eventual aim of this paper: to represent feelings and thereby emotions formally. They do not appear to have a clear logical basis in the same way as the other categories of existence but in category theory we can surmise logical structures based on combining prehensions. Indeed in formal categories the logic contained within locally Cartesian closed categories is in the Heyting logic gate of Figure 6. First the authors look at Whitehead's writings at [P&R pp.23-24]:

(vii) That an eternal object can be described only in terms of its potentiality for 'ingression' into the becoming of actual entities; and that its analysis only discloses other eternal objects. It is a pure potential. The term 'ingression' refers to the particular mode in which the potentiality of an eternal object is realized in a particular actual entity, contributing to the definiteness of that actual entity.

(xv) That a proposition is the unity of certain actual entities in their potentiality for forming a nexus, with its potential relatedness partially defined by certain eternal objects which have the unity of one complex eternal object. The actual entities involved are termed the 'logical subjects', the complex eternal object is the 'predicate'.

Whitehead indicates that the ultimate purpose of eternal objects is their potential for realisation as actual entities. This potential is realised in Figure 5 through their ingression as eternal category-objects B into the relationship $A \times_B D$ where A and B are actual entity category-objects. This is an impure potential as both eternal objects and actual entities are involved. Pure relationships are homogenous, involving only one type of object. A proposition can be more complex than hitherto suggested for Figure 5 with both actual entities and eternal objects having multiple instances, handled through a nexus and a complex object respectively. This does not affect the basic structure of Figure 5, where B for instance can be a complex eternal object and A and D , the logical subjects of Whitehead, can have additional structure, such as a nexus. Whitehead describes the complex eternal object as the predicate. In category theory it is the arrows between the relationship $A \times_B D$ and the complex eternal category-object B that provide the predicate as in the Heyting logic gate $\exists \rightarrow \Delta \rightarrow \forall$ of Figure 6. The diagram $LCCCE$ in Figure 11 shows the elaboration of the topos in Figure 6 with B as ETO and 0 as the unity complex eternal object giving a handle through e to address ETO . Cobb emphasises that the use of the term *proposition* suggests a connection with logic but maybe subsumes it in a bigger picture. In his view Whitehead emphasizes that propositions play a vast role in experience beyond the one they play in logic, but logic may be a good place to begin. ([1] p.46). Later the authors look at propositions in the context of feelings.

Category of Existence	Whitehead's intention	Verbal category view	Formal category diagram	Definition
Actual Entities	Actual Occasions, Final Realities, or Res Verae	Category	Figure 1	CAT
Subjective Forms	Private Matters of Fact	Pullback, vertex only	Figure 4	$A \times_B D$
		Pushout, vertex only	Figure 5	$A +_B D$
Prehensions	Concrete Facts of Relatedness	Cartesian closed category	Figure 4	$CPB \longrightarrow A \times_B D$
	Concrescence with Obligations and Satisfaction	Locally Cartesian closed category with Adjointness	Figure 6	$LCCC \longrightarrow A \times_B D$
Nexus	Public Matters of Fact: Togetherness	Category	Figure 1	CAT
	Defining Characteristic	Pushout	Figure 7	$NEX \longrightarrow DC +_{INH} S$
Eternal Objects	Potentiality for 'ingression' into the <i>becoming</i> of actual entities.	Constant functor. Arrow g is 1:1	Figure 9	$ETO \longrightarrow CAT \times_E E$
Multiplicities	Pure Disjunctions of Diverse Entities	Union with heterogeneous types. Arrow g is N:M	Figure 8	$MULT \longrightarrow DC +_{INH} S$
Propositions	Matters of Fact in Potential Determination, or Impure Potentials ... or Theories.	Heyting logic of locally Cartesian closed category	Figure 11	$LCCCE \longrightarrow A \times_{ETO} D$
Contrasts	Modes of Synthesis of Entities in one Prehension, or Patterned Entities	n-category	Figure 10	n-cells

Table 1. Whitehead's Categories of Existence as Category Theory Constructions

‘being’ ‘entity’. These three notions complete the Category of the Ultimate and are presupposed in all the more special categories.

The term ‘one’ does not stand for ‘the integral number one’ which is a complex special notion. It stands for the general idea underlying alike the indefinite article ‘a or an’ and the definite article ‘the’ and the demonstratives ‘this or that’ and the relatives ‘which or what or how’. It stands for the singularity of an entity. The term ‘many’ presupposes the term ‘one’ and the term ‘one’ presupposes the term ‘many’. The term ‘many’ conveys the notion of ‘disjunctive diversity’; this notion is an essential element in the concept of ‘being’. There are many ‘beings’ in disjunctive diversity.

‘Creativity’ is the universal of universals characterizing ultimate matter of fact. It is that ultimate principle by which the many, which are the universe disjunctively, become the one actual occasion, which is the universe conjunctively. It lies in the nature of things that the many enter into complex unity.

In many respects this is a topos, which can be defined basically as a Cartesian closed category with an initial object providing a lower bound (bottom) \perp and a terminal object providing an upper bound (top) \top . The initial object leads into constructions of coproducts and pushouts (*sums*); the terminal object leads into constructions of products and pullbacks (*times*). The lower bound addresses the basic types used as building blocks in the topos, either through *sums* away from the colimit or through *times* towards the limit. Additional features of a topos are exponentials, ensuring connectivity, and the subobject classifier, requiring the topos to return a truth object. For further details see Lawvere in original description at [12] and a tutorial-style text at [11]; also Mac Lane’s text at [14] has a strong insight into structured categories. Highlighting some areas of Whitehead’s text and our interpretation::

1. “The term ‘many’ presupposes the term ‘one’ and the term ‘one’ presupposes the term ‘many.’”: the relationship between the limit and colimit and their inner components is one to many.
2. “The term ‘many’ conveys the notion of ‘disjunctive diversity’ ”: the sum side of the topos (colimit, coproduct and

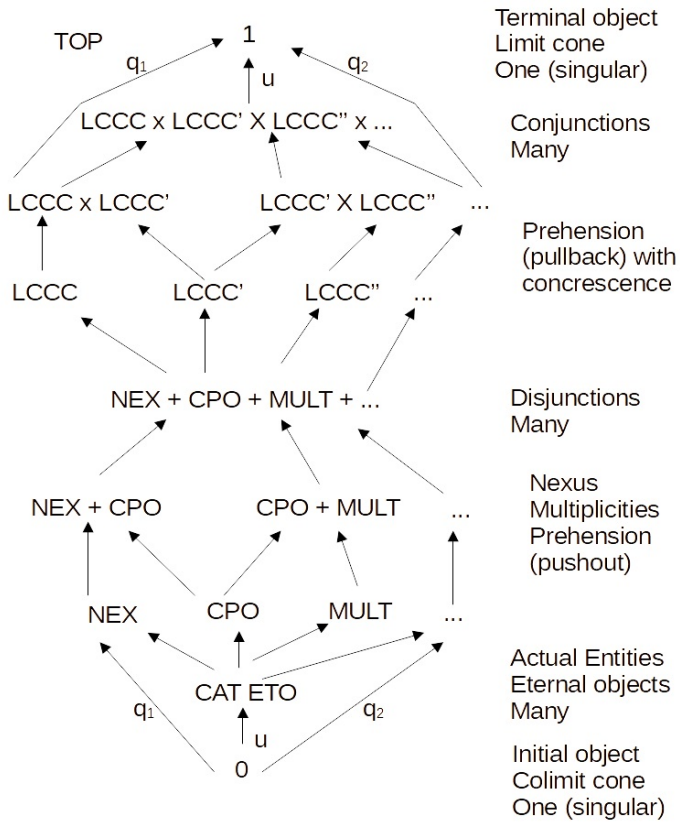


Figure 12. The Topos Category *TOP*: Tension between *times* (prehension) and *plus* (nexus)

pushout) is disjunctive.

3. “‘Creativity’ is the universal of universals characterizing ultimate matter of fact. It is that ultimate principle by which the many, which are the universe disjunctively, become the one actual occasion, which is the universe conjunctively”: the disjunctive coproducts and pushouts are constructed creatively into conjunctive products and pullbacks leading to a single limit.

Whitehead continues at [P&R p.21] with this revealing quote:

The ultimate metaphysical principle is the advance from disjunction to conjunction, creating a novel entity other than the entities given in disjunction. The novel entity is at once the togetherness of the ‘many’ which it finds, and also it is one among the disjunctive ‘many’ which it leaves; it is a novel entity, disjunctively among the many entities which it synthesizes. The many become one, and are increased by one. In their natures, entities are disjunctively ‘many’ in process of passage into conjunctive unity. This Category of the Ultimate replaces Aristotle’s category of ‘primary substance’.

“The advance from disjunction to conjunction, creating a novel entity other than the entities given in disjunction” is close to the process of a topos with the products constructed as novel structures from the sums: the tension between *times* and *sums* in the topos is captured well by Whitehead’s text. The conjunctive unity is the terminal object, the limit cone, of the topos. It is clear that the Category of the Ultimate is an Aristotelian primary substance, that is the extension, holding the data values. Categories, through Dolittle diagrams, can also hold the corresponding intension (definitions) as shown in our earlier work at ANPA on music [21]. Kapinsky [10] also considers that the quote is critical: the complex process, described in the Category of the Ultimate, is, he believed, key to Whitehead’s metaphysics.

Figure 12 shows the topos TOP as a tension between *times* and *plus*. The one initial object 0 maps onto the collection of actual entities and eternal objects $CAT + ETO$. This collection of many objects is then constructed into various nexus, multiplicities and pushouts through union operations. Structures built at one level can be used in higher-level constructions. The result is a

restricted coproduct in disjunctive relationships. This coproduct is then constructed through prehensions into a restricted product in conjunctive relationships as a locally Cartesian closed category. The top-level products map onto the one terminal object 1.

Our use in the example in Figure 12 of locally Cartesian closed categories exclusively on the product side means that concrecence is occurring throughout with the satisfaction of the diagonal adjunctions: the topos is a *becoming* from the disjunctive *beings*. If we used the pullback category *PBO* instead of *LCCC* then concrecence is not assumed and we have the potential for *becoming* rather than an actual *becoming*.

The direction of the arrows in the topos is from the initial object 0 though to the terminal object 1. If we reverse the direction, we get typing arrows as shown earlier in Figure 1(b). Rather than slavishly reproducing the dual of Figure12 we offer first a canonical view of the topos *TOP* as *TOPC* in Figure 13, which shows the flow from the initial object 0 through disjunctions and conjunctions to the terminal object 1 for arbitrary category-objects *A* and *B* in the categories *CAT* and *ETO*.

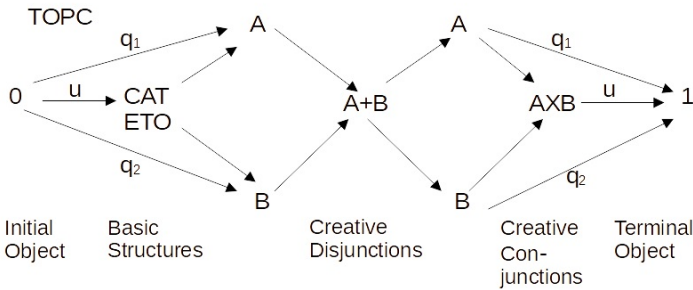


Figure 13. The Topos Category *TOPC*: Canonical Case for Component Categories *A* and *B*

Analogous to Figure 1(b), in Figure 14 we show the dual of *TOPC* as the typing category *TOPC^{OP}* with the object 1 now playing the role of the identity functor. The typing perspective is an important one as this is how proofs and implementations are conducted in mathematics and computing science. The types will include conjunctions and disjunctions, hence prehensions and nexus involving both actual entities and eternal objects.

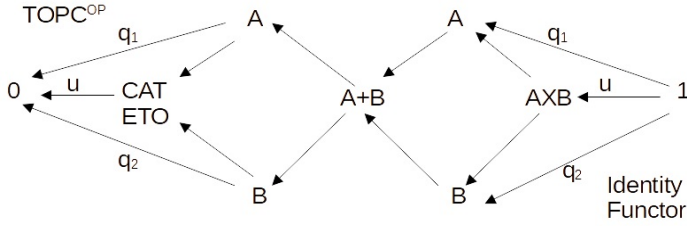


Figure 14. The Typing Category $TOPC^{OP}$: Canonical Case for Component Categories A and B

6 Other Categories

Whitehead also discusses twenty seven categories of explanation and nine categorial obligations. The former expand on his earlier work on the categories of existence. The latter provide constraints on his categorial structures. The authors make some quotes from these sections [P&R pp.22-26 & 26-28 respectively] but at this stage of the work the authors are not dealing with them systematically.

The nine types of categorial obligations [P&R pp.26-27] are: (i) The Category of Subjective Unity, (ii) The Category of Objective Identity, (iii) The Category of Objective Diversity, (iv) The Category of Conceptual Valuation, (v) The Category of Conceptual Reversion, (vi) The Category of Transmutation, (vii) The Category of Subjective Harmony, (viii) The Category of Subjective Intensity, (ix) The Category of Freedom and Determination. These are concerned with maintenance of integrity, performing the underlying functor in adjointness, as the right adjoints in the hyperdoctrine $\exists \rightarrow \Delta \rightarrow \forall$ as described earlier for the locally Cartesian closed category $LCCC$. They play a critical role in concrescence, ensuring integrity in the creative transition from *being* to *becoming*.

7 Feelings

The authors have already considered feelings in the context of prehensions. At [P&R p.211] Whitehead identifies a proposition with a lure for feeling:

The ‘lure for feeling’ is the final cause guiding the concrescence of feelings. By this concrescence the multifold datum of the primary phase is gathered into the unity

of the final satisfaction of feeling. The 'objective lure' is that discrimination among eternal objects introduced into the universe by the real internal constitutions of the actual occasions forming the datum of the concrescence under review. This discrimination also involves eternal objects excluded from value in the temporal occasions of that datum, in addition to involving the eternal objects included for such occasions.

Feelings potentially result from a prehension of an actual occasion (or entity) over the data of an eternal object, termed a conceptual prehension. Since eternal objects include feelings such as senses, moral qualities and qualia, this type of prehension provides a facility *being* for emotion to be handled potentially in the logical structure. The potential of the conceptual prehension is realised as a *becoming* by a concrescence, as described earlier. Such structures are not simple logical justifications but have been described as 'aesthetic valuations', to be employed in wide relevance to existential, ethical, educational, theological, aesthetic, technological, and societal concerns in a Whiteheadian Laboratory [4].

From a data point of view, Whitehead is therefore leading us towards more difficult areas for data structuring such as aesthetics. This is an inevitable consequence of Whitehead's single-substance approach in P&R in which entities of any kind are part of the universe. Some researchers such as Sherburne [25] thought that Whitehead was using terms such as feelings in a technical sense. The authors though concur with Kaplicky [10] who argued that the pervasive use of emotional settings across all of Whitehead's work indicated that Whitehead's metaphysics was inherently underpinned by aesthetics through the use of terms such as feelings and senses in his description of entities and objects. While feelings might be thought to be outside the normal realm of data types, there are developments in artificial intelligence, using facial recognition or voice analysis techniques, that claim to be able to recognise feelings such as happiness, sadness, fear, anger, disgust and surprise. Pheromones are another area under investigation, with feelings associated with love and fear, for example, potentially transmitted through odours. So it is possible for new data types to be created in this area with appropriate processing operations for handling emotion. Whitehead himself anticipated this route as he stated that each feeling has a physical component. He recognised negative prehensions as well as positive ones: the negative prehen-

sions appear to imply in categorical terms the breakdown of logic rather than negative feelings such as sadness.

8 Discussion: Philosophy of Category Theory

The authors start by introducing Rosen's terminology into Whitehead's work: is Whitehead's a natural or a formal system? It is not a model or a formal system. Indeed it might be better described as the causality arrow ① on the natural system, examining the mechanism by which an entity develops through prehension. So with Whitehead ① is prehension and ②-④ are undefined. On the other hand our category theory system is formal so arrow ② gives the encoding from the natural system to our formal system, arrow ③ establishes the implications of the formality, arrow ④ gives the decoding from the formal system to the natural system. Figure 15 shows the Rosen diagram of Figure 2 in more typical category theory style with adjointness $④ \dashv ②$ between decoding and encoding to reflect imperfections of the mapping between reality and formality with ④ the free functor and ② the underlying functor.

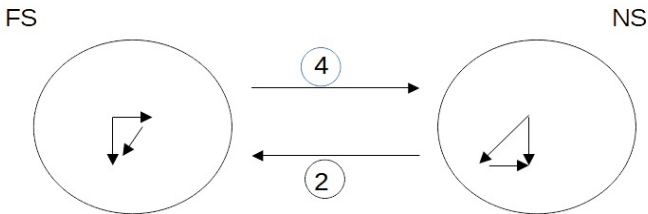


Figure 15. Adjointness between Rosen's Formal System FS and Natural System NS : $④ \dashv ②$

The mathematical basis for category theory is underpinned by the dominance of arrow over object. The arrow retains though in the main a rather limited interpretation based on small categories (set-like constructions) as this facilitates mathematical proofs utilising a well-known type system. Large categories, which do not rely on sets, show more potential in new areas such as biology but are relatively unproven and the question will be: what are your type systems? Homotopy has been proposed as a broader advance in range of application areas [27]. Homotopy allows continuous

functions, hence the real numbers, to be handled. But the difference between a fine integer type system and a real type system are minor. In our application of category theory to music [21], we mentioned for the violin the range of microtones utilised in modern music, going beyond the limitations of the 12-tone scale. However it is unlikely that microtones will be continuous, except perhaps for AI-generated music, so a finer scale may well suffice in a persisting discrete system such as Partch's 43 tone-scale [18].

The authors have always tried to avoid categorification, a simple-minded translation 1:1 of concepts in the target application into category theory. It should first be said that for Whitehead's work it is not possible to understand all of the text as he writes in an introspective way, which is almost idiosyncratic at times and without mathematical clarification. So it would not be sensible to translate Whitehead's ideas expressed in great detail in words into any formalism with great precision. The authors have concentrated on the main thrust of Whitehead's work, which appears to focus naturally on Cartesian closed categories and the topos.

In computing science implementations of basic types has evolved from bits to bytes to numbers (integer, reals) to dates to characters to text to images to video over the past 70 years. In object-based systems such as Java the abstract (user-defined) data type was developed with arbitrary internal structure and operations appropriate to the type. An abstract data type is encapsulated with internal structure and processes hidden from applications, which can only access the type through a predefined interface. In mathematics type is broadly synonymous with category but refinements have been sought to handle aspects such as extensionality. Foremost in mathematical developments is the Martin-Löf intuitionistic type theory [15], based on Heyting logic. Inductive types allow the creation of complex, self-referential types, rather like abstract data-types in computing science. Extensional data types are more useful in practice because they handle data as well as definitions but lately the move has been to intensional types, as they are more amenable to proof. This is analogous to the tension between large and small categories.

The Heyting logic is the logic of the topos so the diagrams earlier in this paper such as Figure 12 is a type, indeed an extensional one as defined by Whitehead. There does appear to be a framework in the topos diagrams in this paper for both Whitehead's Category of the Ultimate and for a theoretical type-system for handling feelings. The realisation and testing of such a type system relies on

advances in machine-based sensory perception and artificial intelligence to process and analyse feelings. In conclusion Whitehead's category system, based more broadly on process rather than the mathematical arrow of pure category theory, offers a step forward in ambition.

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