Physical Sounds as Colimits in the Topos

Nick Rossiter
Visiting Fellow
Computing Science
Northumbria University

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Liverpool University

Acknowledgements

- Michael Heather
  - Work presented here builds on our earlier work with the topos and the monad
  - Our earlier analytical work on time jitter in categorial terms (Liege) has proved to be relevant
- Musicians who have inspired my interest in the art and helped me to understand it

The Classical Topos

- The topos is based on the idea of Aristotle for tackling a legal argument with the premises held within its structure and the logic returning true or false as the outcome.
- In category theory the classical topos as defined by Grothendiek and others is closed at both ends and the truth object or subobject classifier may be more complex, for instance based on the natural numbers.

Times and Plus

- At the lower end there exist products of objects, connected by times X, and a limit.
- At the upper end there exist coproducts of objects, connected by sum +, and a colimit.
- Limit is greatest lower bound
- Colimit is least upper bound
- Interplay between X and + plays a critical role in categorial applications
- This is the Cartesian World

Real-world Topos

- If either the limit or colimit does not exist, then the category is not a classical topos.
- The existence of the colimit requires a single (unique) arrow from it to every object in the topos
- If the arrow is not unique then the colimit is said to be weak and the structure is not a topos
- We are exploring this condition in our work
- The colimit is the initial object of the topos

Example of Music 1

- Recent work by the authors has concentrated on the example of music in category theory
  - logical aspects such as
    - players and scores
    - occasions, representing a co-ordinated sound by the performers
    - administration of the concerts
- Performance has been of particular interest
  - Monad as process represents movement from one timeline to another

Example of Music 2

- Monad IA emulates musical processes in the brain for
  - Adjunction <A, I, η, ε>
    - A (Articulation) is free functor (lhs of brain)
    - I (Intonation) is underlying functor (rhs of brain)
    - η, ε are unit/counit of adjunction for offsets in mapping
  - Example is clearest for violin with
    - Bowing by right-hand feeding into lhs of brain
    - Finger control by left-hand feeding into rhs of brain
- Monad takes one (adjoint) step through the timeline
- Also dual Comonad AI

Published

- Rossiter, Nick, & Heather, Michael, Musical Performance: a Composition of Monads, Sociology and Anthropology 7(4), 178-188 (2019)
- Fuller version submitted to ANPA 39

Physical Sounds as Categories

- This paper will take the work forward by exploring how
  - physical sounds are defined in terms of a topos
  - the conditions for colimits to exist
  - the boundaries of their existence in the context of an actual performance

Emphasis

- Always interested, as computer scientist, in computer applications of category theory, particularly in Haskell.
- Concentrate on re-use of categories. Avoid pasted structures which reduce modularity.

Music Genre

- Giving plenty of thought
- Music is often viewed as discrete
  - through keys, notes, named chords
  - as incorporated into scores
- But the physical sounds are waves with amplitude and frequency, associated with a pitch in Hz
- Chords have complex physical properties (harmonics), particularly when overtones are considered

Examples of Genre 1

- Popular music is the simplest form with generally no dissonance, all notes within a particular key, low range of pitch
- Classical music is much more varied than some people think:
  - Strays readily outside tonality (diatonic scales)
    - Chromaticism (foreign notes, not in diatonic key)
    - Use of chromatic 12-note scale throughout with only semitone intervals
    - Much dissonance
    - Individual experimentation: e.g. Ligati (every player has a different score), Chopin (rubato, expressive tempo)

Examples of Genre 2

- Jazz
  - Improvisation, some written score but much freedom of expression
- Film music
  - Links to drama, can be austere, disturbing
- Chanting
  - Very precise attention to the beat
- Microtones
  - Intervals less than a semitone are used
    - For freedom, easier on string instruments
    - For music from diverse cultures
      - But discrete patterns may still be identified

Consequences

- The work presented here is suited to popular music, taking a simple discrete approach
- It can be readily adapted to the 12-note scale as still discrete
- However, did consider handling microtones, including Stockhausen's music, which is reputed to be amenable to category theory.
- This requires a move from discrete to continuous maths.

Continuous Mathematics as Categories

- Real numbers
- Vector spaces
- Tensor products
  - Not Direct Sum of Cartesian, tighter product
- The Ring Category, based on the Abelian Category (dual of a Dolittle diagram with coequaliser followed by equaliser)
- The work of Peter Freyd is relevant
  - claims that his work is equivalent to the discrete topos approach but with different initial object (colimit)

Thoughts on the Ring

- In progress!

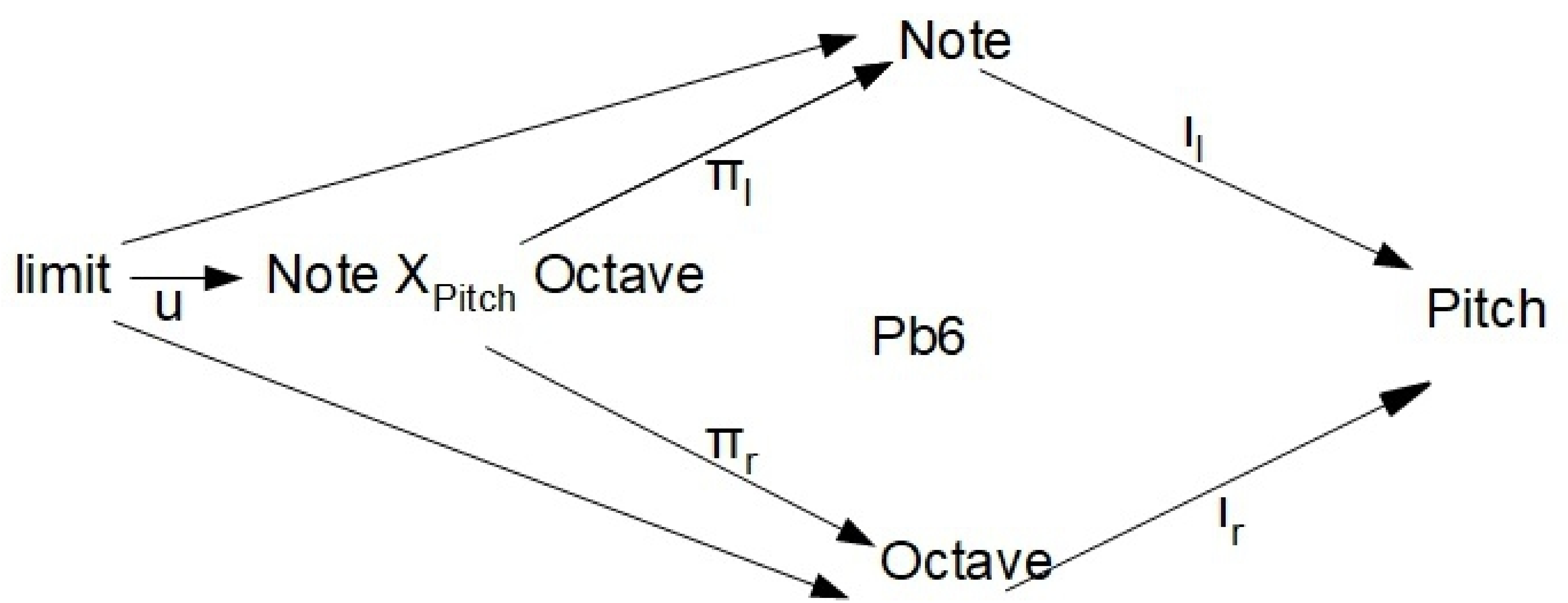
The Whitehead 'Now'

- How do occasions relate to the `Now' of the philosopher Alfred North Whitehead.
- Every entity in the World has a fleeting (covariant) capacity to act on and a (contravariant) perception to receive from its immediate surroundings.
- That is a local `now’ in both time and space giving rise to the synchronicity of the actual occasion where all the separate loci of each entity meet.
- These local `nows’ all compose to form the big NOW

Overall Aim

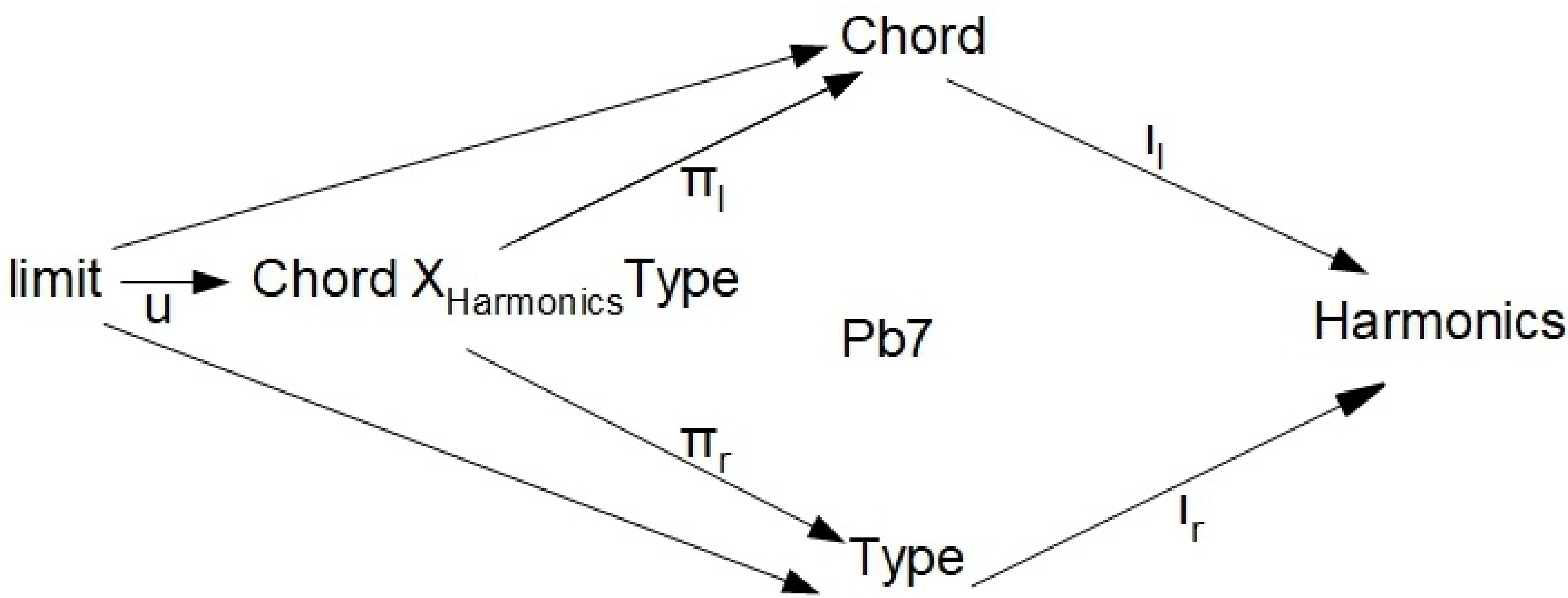
- The result should advance
  - Our understanding of music in universal terms across both the logical and physical levels
  - Feed back into category theory a better understanding of the formation of colimits
- Worth noting that music is seen as an example of human communication

Relationship Pb6: Note over Octave in context of Pitch: Permissible Notes



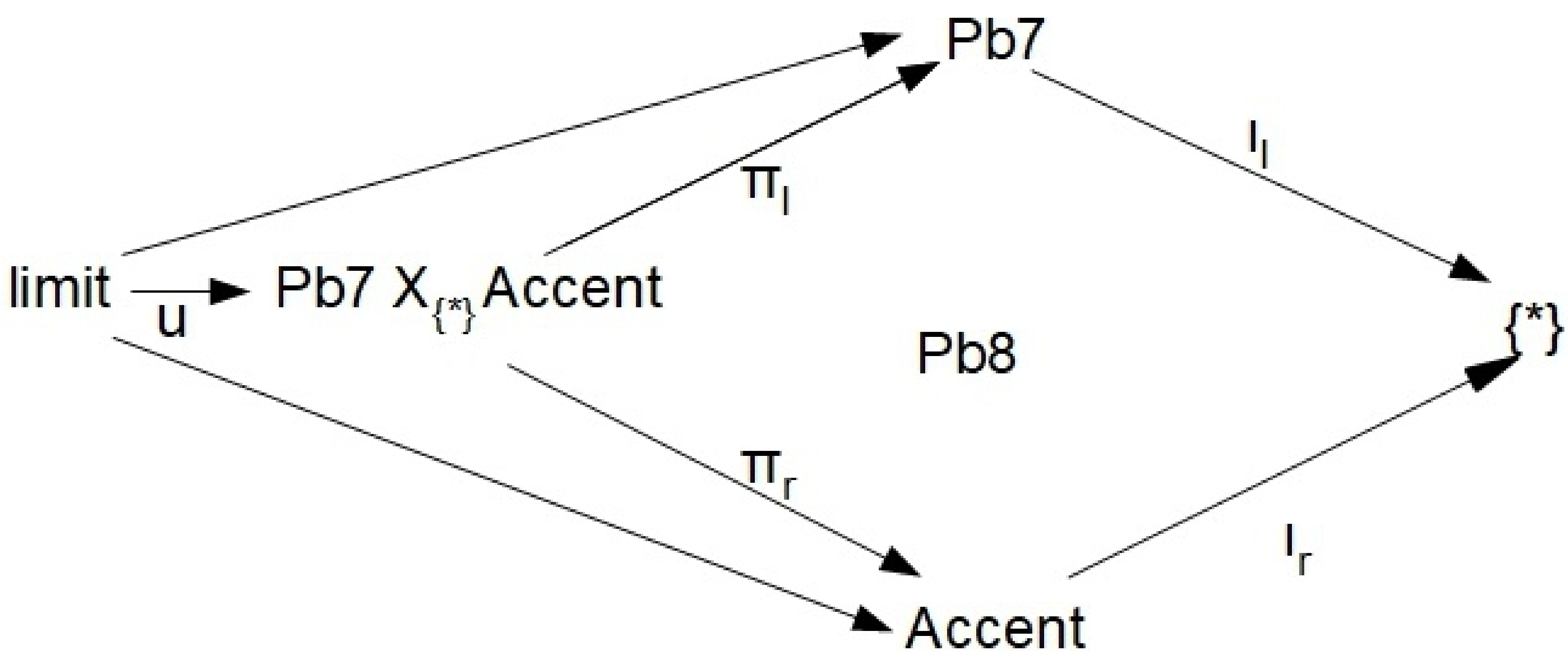
Example: Note = C, Octave = 4, Pitch = 261.626 Hz (middle C)
Note X_{Pitch} Octave = <C, 4, 261.626>
Typing: can restrict e.g. Note = {c,d,e,f,g,a,b} for C Major scale; Octave = 0-7

Relationship Pb7: Chord over Name in context of Harmonics: Permissible Chords



Example: Chord = C,E,G; Type = C major; Harmonics = root, major third, perfect fifth
Chord $X_{Harmonics}$ Type = <C,E,G; C major; root, major third, perfect fifth>
Chord is a powerobject of Pb6, all possible combinations with Note already typed
Chord may hold 1,2, 3 or more notes

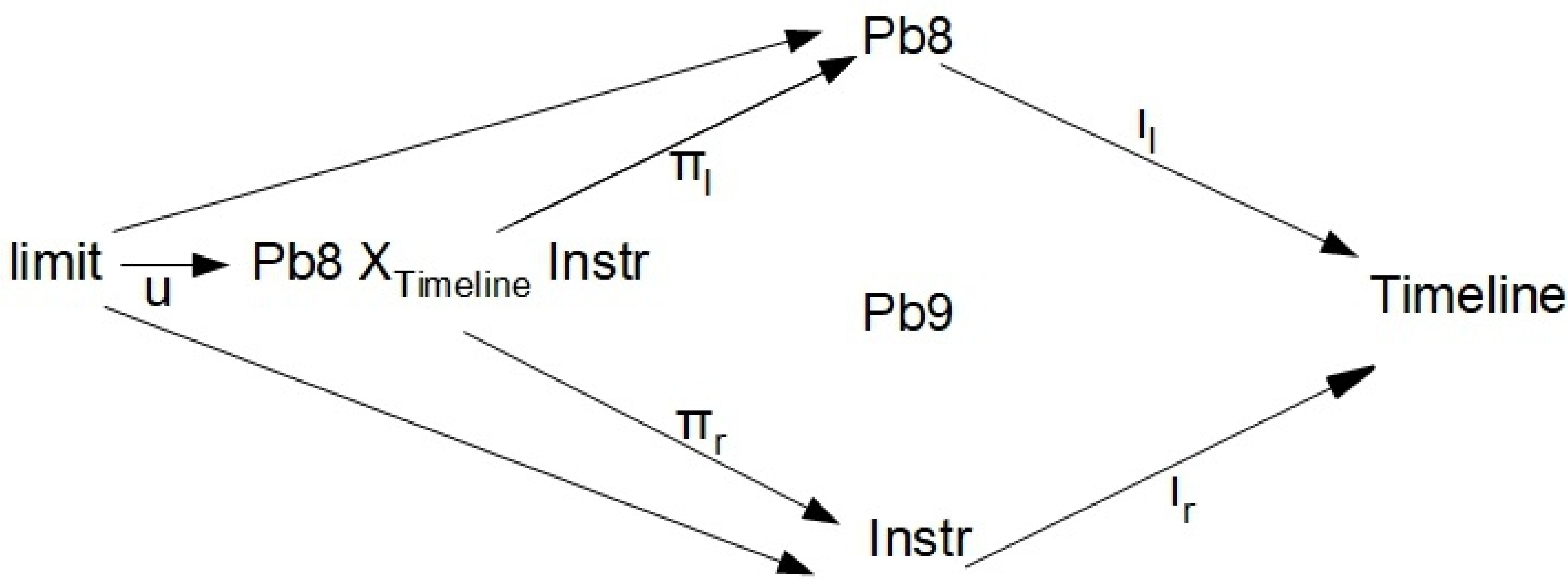
Relationship Pb8: Pb7 Permissible Chords and Accents



Example: Pb7 is an instance of a permissible chord, say C major; Accent = staccato;
{*} is universal object, giving unconditional product
Pb7 $X_{\{*\}}$ Accent = <C major, staccato, {*}>

Relationship Pb9: Pb8 Permissible Accented Chords with Instrument over Timeline (the theory)

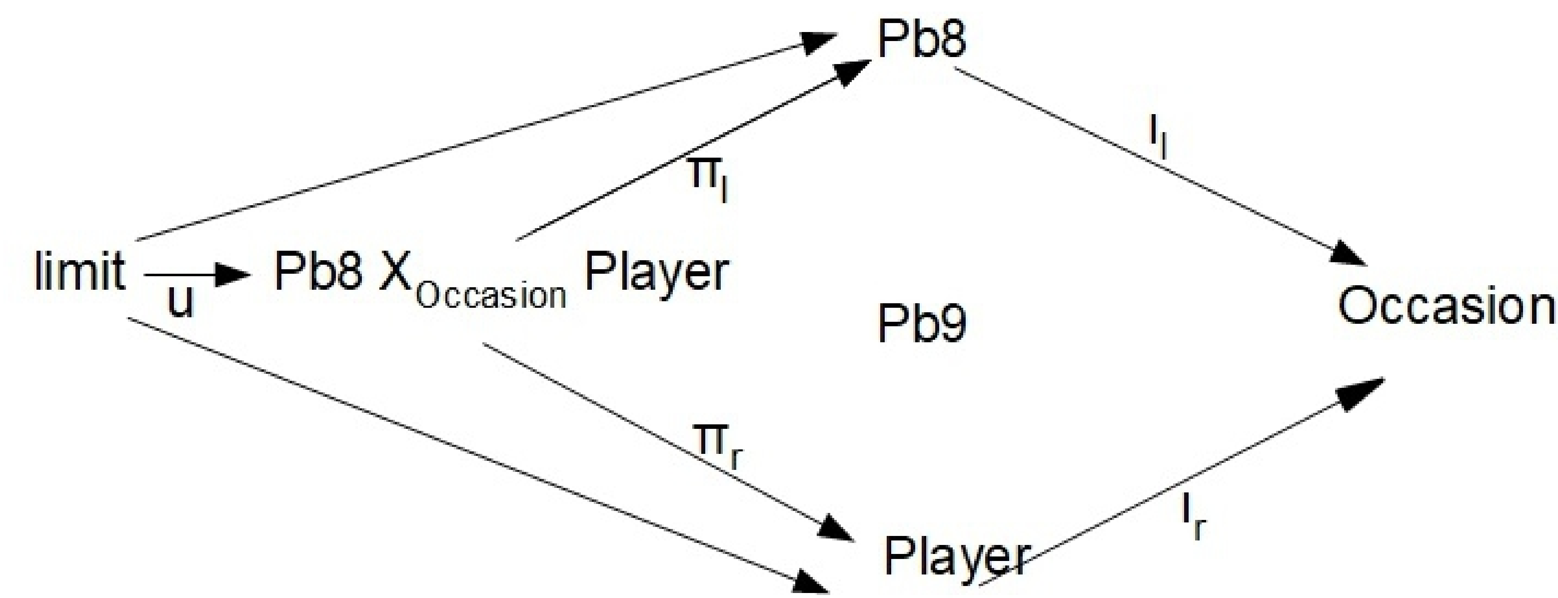
INTENSION



Assigns a permitted accented chord, a collection of note(s), to an instrument for a Timeline, that is a particular position in the score, bar+offset, together with length of note e.g. bar 117, offset 2, length 1, in say 4/4 time

Relationship Pb9: Pb8 Permissible Accented Chords with Player over Occasion (as played)

EXTENSION



Occasion is the performance of the accented chord by the player of the designated instrument

This diagram is the extension of Pb9 as intension, with label changes of Player for Instrument and Occasion for Timeline

Timeline:Occasion is 1:N

- There are many occasions for each timeline:
  - 1 for each instrument
  - So Pb9 extension is a canonical case for the Occasion
- But only 1 timeline
  - Only one Pb9 intension

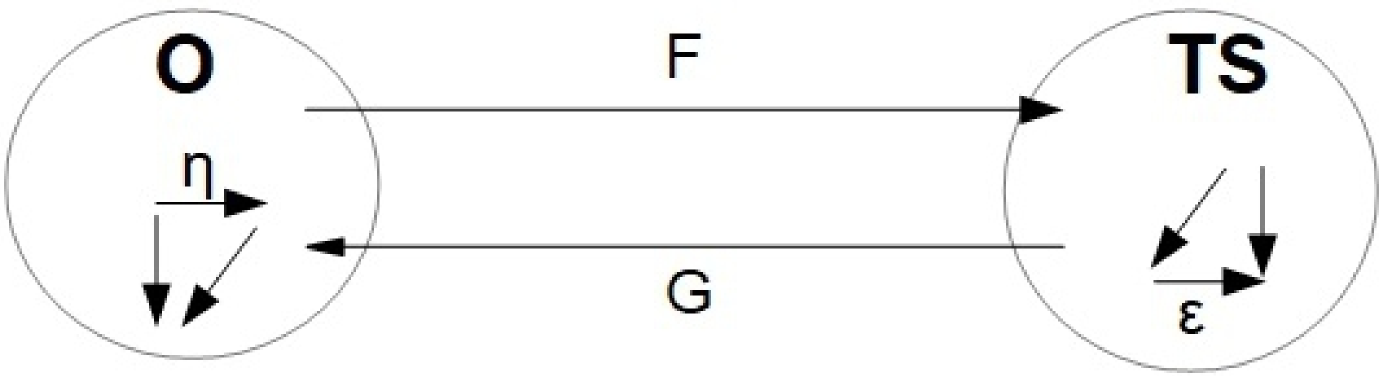
Adjointness

- If the players are perfect (Berlin Phil, von Karajan!) then this is the complete picture as every Occasion matches the expected Timeline
- Strictly the values for the extension should be in accordance with the rules of the intension so there is an assumption that all the players are perfect.
- However, there is a type issue: we should compare the sound of the intension (not the score) with the sound of each individual player and consider jitter

Adjointness Timeline_Sound/Occasion

- Timeline_Sound is the anticipated sound produced from the Score
- Occasion is the sound actually produced by a player
- Functor F takes Occasion (O) to Timeline_Sound (TS)
- Functor G takes Timeline_Sound (TS) to Occasion (O)
- We can then look for adjointness GF with unit/counit η, ε for offsets in mapping

Adjointness between the mappings from Occasion to Timeline_Sound (TS)



F --| G, F is left-adjoint to G, G is right-adjoint to F
Adjointness is written <GF, η, ε>
F is free functor, G is underlying (forgetful) functor

Unit of adjunction η
Counit of adjunction ε

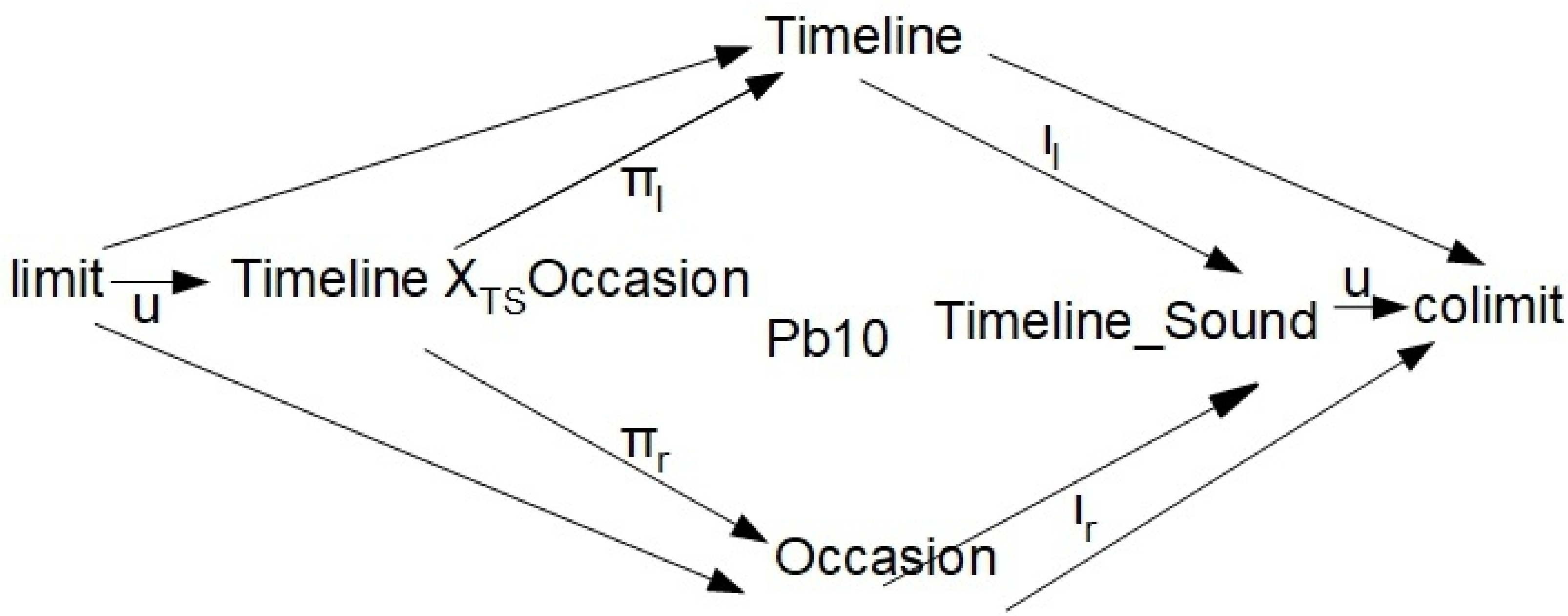
Adjointness Variability

- The adjointness will not be the same for each pair of Occasion and Timeline_Sound (TS)
- Different players will have slightly different approaches
- So we have for example:
  - F: Occasion → TS, G: TS → Occasion
    - $<GF, \eta, \varepsilon>$ (flute)
  - F': Occasion' → TS, G: TS → Occasion'
    - $<G'F', \eta', \varepsilon'>$ (clarinet)
  - F'': Occasion'' → TS, G: TS → Occasion''
    - $<G''F'', \eta'', \varepsilon''>$ (bassoon)

Jitter and NOW

- Variability of players gives rise to jitter, slight 'trembling' along the timeline.
- Need to look at asynchronous network communications for some guidance
- The adjointness between each instrument and timeline_sound can be composed, in the same way as other arrows, to give the big NOW.

Dolittle Diagram providing Colimit for Topos: Canonical Case Pb10

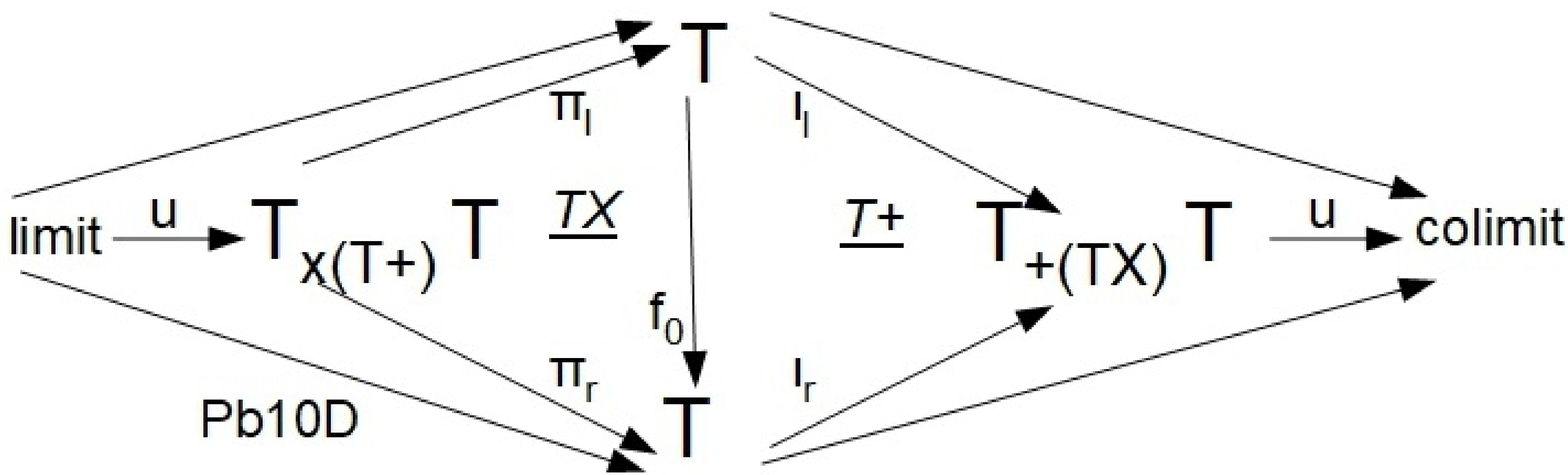


In general a pullback has a limit but no colimit.
When the pullback is a Dolittle diagram as in relating intension to extension, then it is also a pushout giving rise to a colimit on the + side
Intension is Timeline, extension is Occasion
The colimit is the least upper bound of the topos, which has all products and coproducts
There is a set-valued functor from Timeline to Occasion

Dolittle Diagram providing Colimit for Topos: Categorial Object Pb10D

T is category Timescale

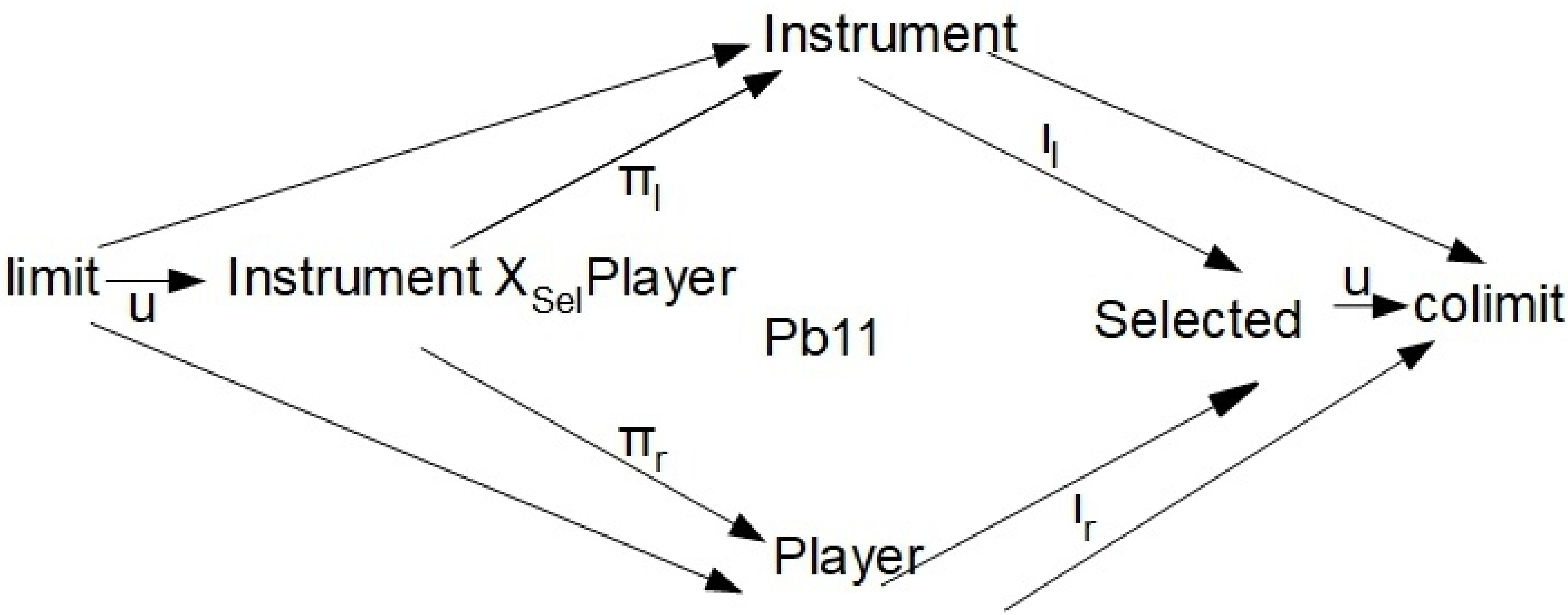
The set-valued functor from Timeline to Occasion is f_0



Source of f_0 is T as intension (Timescale)

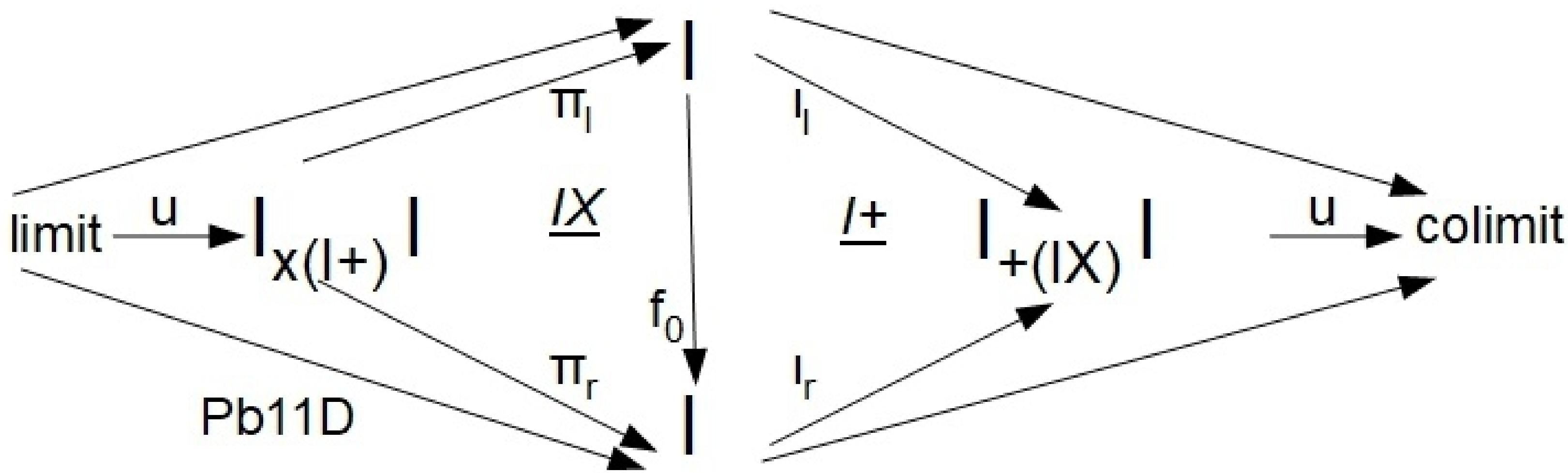
Target of f_0 is T as extension (Occasion)

Dolittle Diagram providing (local) Colimit for Instrument: Canonical Case Pb11



Dolittle Diagram providing (local) Colimit for Instrument: Categorial Object Pb11D

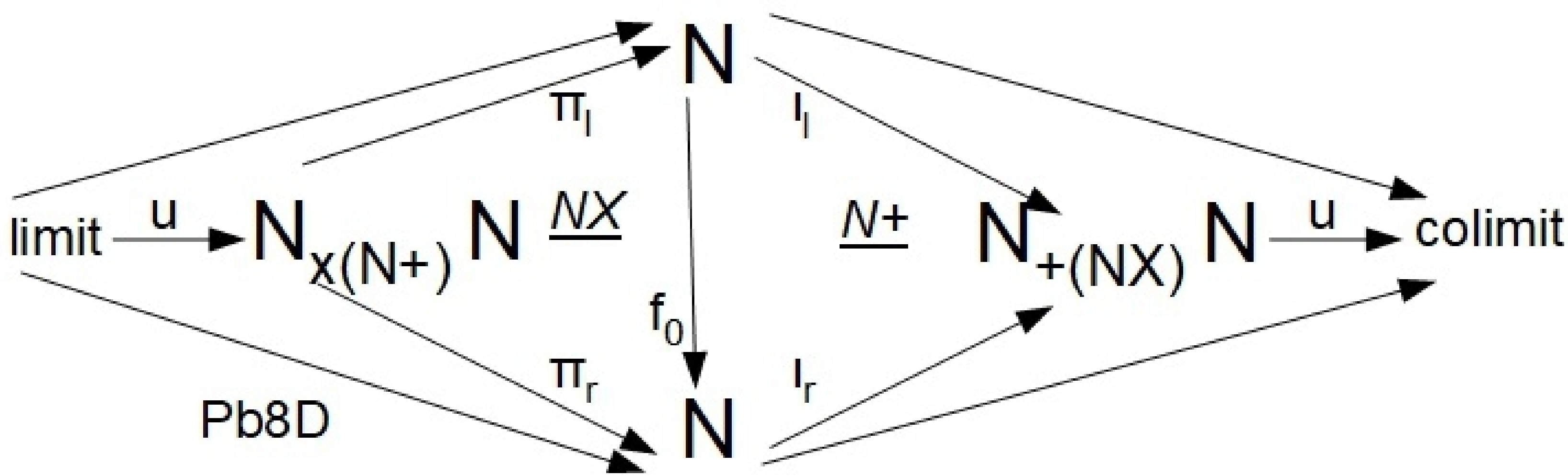
I is category Instrument
The set-valued functor from Instrument to Player is f_0



Source of f_0 is I as intension (Instrument)
Target of f_0 is I as extension (Player)

Dolittle Diagram providing (local) Colimit for Permitted Accented Notes: Categorial Object Pb8D

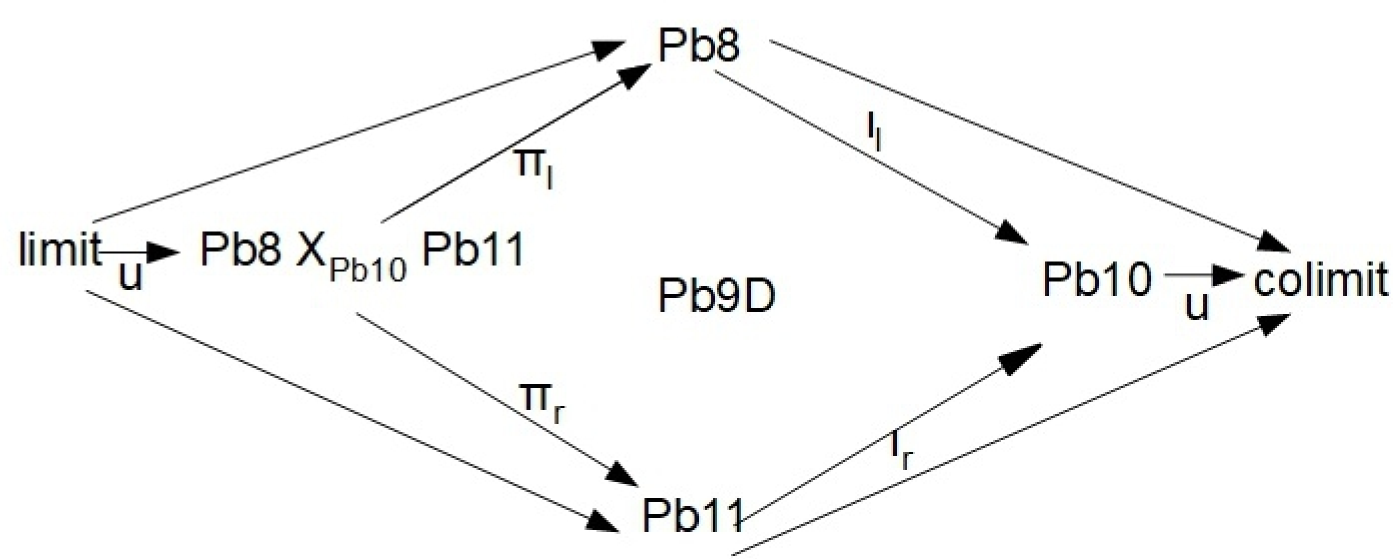
N is Permitted Accented Notes category, a type definition
The set-valued functor from the type definition to the sets of permitted notes is f_0



Source of f_0 is N as intension (type definition)
Target of f_0 is N as extension (permitted notes)

Relationship as Dolittle Pb9D: Pb8 over Pb11 in context of Pb10

INTENSION+EXTENSION (Global Closure for Topos)



Pb8 is a permitted accented chord
Pb11 is instruments with their players
Pb10 is the timeline of the score with occasions as played

Pb10 is the Conductor (Maestro)

- It is his/her task to realise the Timeline (intension) as the Occasion (extension) for every instrument
- What is realised can be compared to the ideal sound expected
- There may be style or performance reasons for differences
- So the Conductor can be viewed as the initial object of the performance

Observations

- Music is proving to be a fertile area for the application of category theory
  - The topos structure for discrete scales is verified
  - The colimit is the sound from the timescale
  - Individual adjointness for each player's performance can be represented
- The handling of music with continuous maths is a major challenge
  - But potentially very rewarding from the physics point of view