Locality, Weak or Strong Anticipation and Quantum Computing. II. Constructivism with Category Theory

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1 Constructivism and the Quantum Computer

Constructivism in mathematics concerns existence and proof. That is proving that a mathematical object exists and reaching conclusions to demonstrate connections between such objects not just by verification but by construction. But where do the mathematical objects and their connections exist? Methods adopted by pure mathematicians suggest the answer that they are in the mind. It is an ontology of the mind: mathematics operates at an epistemological level. This has been very successful even in applied mathematics where mathematical models are epistemological representations of real-world systems. With anticipatory systems the phase change is not usually between reality and mathematics but between one mathematical representation and another: a reactive system and its anticipation [46]. (This is an important distinction between models and anticipatory systems. For anticipatory systems are not just models by another name). Applied mathematics therefore mostly make do with non-constructive forms. This is even true for topics beyond the classical range like quantum theory so long as we are only concerned with epistemological models. This has been the history of the development of quantum theory with its many interpretations. One finds non-constructive proofs at places throughout mainstream quantum theory. For instance the indistinguishability of non-orthogonal quantum states is proved by Nielsen & Chuang (in Box 2.3, [43]) by a method of *reductio ad absurdum*. Also Bell's theorem is local and involves a non-constructive proof. The concept of the *qubit* is derived from this case of a quantum object by analogy with a binary digit (a *bit*). In the terminology of category theory the term *a quantum subobject* would seem to be preferable. Part/whole complementarity is a pervasive theme of non-locality [41] and can be well represented in category theory whereas set theory is very restricted because a set cannot be a member of itself.

Now however we are talking about building systems with significant operation in the quantum zone. This is ontological construction and requires even more than mathematical constructivism as currently envisaged. For with quantum computers we are more concerned with hardware than with software.

Computation itself is a very good example of this phenomenon at the epistemological level with a vast amount of thinking subsumed under the notion of the universal Turing machine. The Turing machine is more of an abstract software machine than a piece of hardware. That is it really only exists in the mind. Davis is able to survey the history of computation [11] from Leibniz to modern computers without the need to acknowledge the point. However, Deutsch [12] in his seminal paper showed that the principle of universal computation as found in the Church-Turing hypothesis could be extended to its quantum analogue. However the Church-Turing hypothesis and the quantum theory used by Deutsch is still only at the epistemological level. As such it may be a very good model and an anticipatory (knowledge) system for the quantum computer but only with weak anticipation. Strong anticipation is needed to construct a real quantum computer.

The salient point is therefore that a digital computer can perform any calculation of a universal Turing machine. According to Turing: "Logical Computing Machines can do anything that could be described as 'rule of thumb' or 'purely mechanical' " [56]. Church's phrase for 'purely mechanical' was 'effectively calculable' and now following Gödel this is more specifically referred to as recursive functions (of positive integers, it is to be noted). This is not the same as anything computable by the human brain or more appropriately here computable by the universe or part of it. We have to conclude therefore that the universal Turing Machine is only a weak anticipatory system. The quantum computer is on the other hand a strong anticipatory system. This suggests that the Deutsch specification for the quantum computer [12] as only epistemological is an inadequate ontological basis for the construction of a physical quantum computer. We are in the realm of constructive applied mathematics which may not correspond exactly with what is commonly referred to as constructive mathematics.

How does all this relate to logic? The basis of computation is logic. Classical logic provides weak anticipation. As a weak anticipatory system the universal Turing machine can operate with classical logic and under the Church-Turing hypothesis this is adequate for classical computers. By a parallel strand of argument the standard model description of quantum mechanics is a weak anticipatory system. However, as pointed out by Landauer [35, 13, 14] computation is ultimately a physical process and quantum computation is the ultimate process. As a part of the Universe, quantum computation is a strong anticipatory system. The underlying logic has therefore to exist in physical reality. That is, it is constructive logic.

Mathematicians have explored constructive logic in the context of intuitionistic reasoning. Brouwer won the argument for his intuitionistic form of mathematics (potentially a strong anticipatory system) against Hilbert's programme to establish classical mathematics formally [28]. Hilbert lost the argument that physics can be axiomatised [10] but won the day in establishing classical mathematics as a mainstream model for science. This was because it is, as far as it goes, a weak anticipatory system even if rather inefficient. Now with the advent of the strong anticipatory system of quantum computation we need a correspondingly strong anticipatory form of logic. Does intuitionistic logic satisfy the requirements of a strong anticipatory system?

Brouwer's intuitionism [55], Markov's recursion analysis [34] and Bishop's constructive analysis [9], these different schools all have a common basis of logical reasoning. They allow the law of contradiction (that anything follows from a contradiction) but not the law of excluded middle (*tertium non datur*). Brouwer's informal intuitionistic mathematics in the hands of his student Heyting [26] proved to be, not less but, more formal than classical mathematics. The philosophical aspects are dealt with by Dummett [16] but unfortunately for present purposes the various treatment of intuitionistic logic is still couched very much in weak anticipatory terms. This is because they are normally considered in the context of pure not applicable mathematics. Bishop for instance relies heavily on the fundamental notion he calls 'finite routine' but does not define this in any applicable sense. Intuitionistic logic can perhaps be said to show the way but cannot be relied upon conclusively as a strong anticipatory system. Natural language may have the power of strong anticipation. Aristotle, usually credited with the invention of symbolic logic, relied more heavily on natural language expressions and went further even than the modern intuitionists in examining the fine structure of the copula 'is'. Thus in the Organon Aristotle distinguishes the truth value of *Socrates is not ill* which is true even if Socrates does not exist from that of *Socrates is ill* which is not true if Socrates does not exist (*Categories* $13^b15 - 35$ [1]). Aristotle does not go on to consider the corresponding truth value of *Socrates is well* but it is to be presumed he would treat that as not true if Socrates does not exist and so reject *tertium non datur* in the copula. It is bridging the gap between the weak and the strong anticipatory system that we contend is supplied by the category theory [24], where intuitionistic logic has an incarnation in physical reality comparable with natural language as illustrated by Aristotle's examples. The Philosopher himself however did not apply this logic to physics although he seems to have gone further down the intuitionistic road than is usually credited.

2 Applying Non-local Category Theory to Quantum Theory

The physics of quantum is process: Aristotle's was a physics of types and Newton's one of primary properties. Jammer ([30] p380) quotes Høffding: "The 'qualities' of a thing are indeed nothing more than the different forms and ways in which this thing influences that thing or is influenced by it. They are a thing's capabilities of doing and suffering" [29]. This sums up the covariancy and contravariancy of nature. Jammer continues ([30] p381) with his own view that "the language of quantum mechanics is a language of *interactions* and not of *attributes: processes*, and not *properties*, are the elements of its syntax". These are descriptions of the categorial arrow and seem sufficient reason for the use of category theory, but there is much more: it is constructable in a mathematical sense and does not require *tertium non datur*. It can represent both right- and left-class in Table 1 in Part I of this paper. It can deal with concept of choice and free will (this is latent in Figure 1 in Part I of this paper). It is not constrained to use any particular reference coordinate system, 'container' or background for describing entities like time and space.

To begin with a given container like space-time is to pass from strong to weak anticipation. These are consequences of theory and cannot also be given *a priori*. We cannot assume initial frames, coordinate systems, etc. Since Einstein's theory of special relativity it is not just space or time but also space and time that is space-time. However, also since Einstein's general theory, neither space nor time nor space-time are independent of matter. Matter does not exist in a space-time container [42]. The matter makes the container. Observations of the relationship between matter is what we call motion. It does not exist with respect to any 'background' (Rovelli [49, 50]). The relationship between motion (namely acceleration) by Mach's principle corresponds to mass. This is where category theory comes into its own as a geometric-kinematic representation rather than one like topology which is geometric-spatial.

There are always problems at the foundations of mathematics and care has to be taken to ensure that the category theory applicable to realising actual quantum systems in the real-world has robust foundations. Over the last two centuries mathematics has developed with the emphasis on axiomatic methods. Because these are epistemological and derived by filtering through human intuition (which may itself be a quantum process in consciousness) these axiomatic systems have been fairly successful. Yet this is not always the case and as we have no *a priori* scientific basis for any axiomatic system we cannot be sure of such foundations. Category theory is usually presented in text books axiomatically and from a set theoretic perspective with the use of objects. However, the whole of the (pure) theory only requires the concept of the arrow. The arrow represents the (applied) theory of the universe as a process.

It can be shown that a version of Zermelo-Frankel set theory with the axiom of choice (but independent of the continuum hypothesis) is a valid model within a general elementary topos (Mac Lane & Moerdijk, Chapter VI) but with no distinction between 'global and local existence' ([38] Chapter VI, Section 10). However, this is weak anticipation. Boolean and Heyting algebras are not isomorphic.

We need to be non-local and to be constructive. Weak anticipation in the form of local theory lies in the realm of epistemology which may be important for understanding quantum theory, quantum computing and quantum information systems but is inadequate for constructing quantum computers. This requires a move into strong anticipation and the realm of ontology. Historically category theory has developed in local mode using non-constructive proofs and with an emphasis on the category of sets. This can often give better understanding of set-theoretic models. For it is possible to use n-categories to model quantum field theory [2, 3, 4] or in 2-categories to describe a more advanced categorial analysis of the Hilbert space [4]. However, these are still local methods and we cannot be sure therefore we can rely on these versions of category theory any more than we can rely on set theoretic methods to design and build a quantum computer. For using category theory as an anticipatory system of existing mathematical models is a two-stage process and can be expected to provide only a better understanding and not more information. Whereas the direct application of category theory offers the opportunity for strong anticipation by constructive methods.

The fundamental constructor is the concept of the arrow. The universe is just one single arrow [25] consisting of composable arrows in the sense of Figure 1.

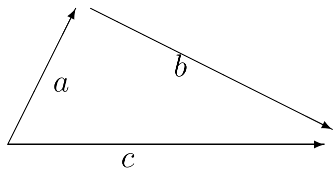

Fig. 1: Composition written as $c = b \circ a$

The diagram in Figure 1 is itself just one arrow, the composition of b with a written $c = ba$ by convention where b operates on the result of the operation a. Diagrams are formal statements but constructive. This diagram is a proof of the equivalence (or the quality, depending on context) identifying c with ba. However, the categorial version is more formal than the algebra for it will not permit statements like $c = ba$ without defining what is meant by equality. If ba is indistinguishable from ab this diagram is an identity arrow. That is, it just identifies its own existence and for convenience is usually referred to as an object. Categories are composed of objects (that is identity arrows) and other arrows relating them.

For applied category theory the principal (possibly the only) category of interest is the one with co-limits, that is with identifiable existence and known as the cartesian closed category. The name is not to be confused with a cartesian coordinate reference frame and the epithet 'closed' does not mean that it cannot have the property of openness. Some concepts of the cartesian school of philosophy are embedded in the concept of the cartesian closed category, for example the denial by Nicholas Malebranche that like entities exist [37] is borne out in the difficulty of defining 'equality' as mentioned in the last paragraph. Another example is the principle of Spinoza that the infinite is contained within the finite (Spinoza's letter, *On the nature of the infinite*, to Lodewijk Meyer, 20th April 1663 [51]). That same point was a prime motivation for the development of category theory by those like Mac Lane [39], the co-author with Garett Birkhoff of the main student text on algebra [8]. If John von Neumann had collaborated with Saunders Mac Lane rather than Birkhoff the development of quantum theory in the second half of the 20th century may have been much accelerated by the application of category theory. Instead the general notation used is that of Dirac in his monograph [15], which has been continuously in print since 1930 because of its quite elegant notation. It is nevertheless quite idiosyncratic, rather obscure and lacks the clarity of category theory with its abstract universality, inherent logic and natural ability to express global non-locality.

Because a cartesian closed category has co-limits it also has limits like exponentials, usually written in the form Y^X meaning all arrows from object X to object Y. This describes the universe as what is accessible from one another. The arrows of Y^X themselves form a category written as $C(X, Y)$ or $hom(X, Y)$ in the old terminology from homology. The universe is therefore a category of categories which is a topos. A significant feature of the topos is that its internal logic is the intuitionistic logic of Heyting [7, 31, 38].

Cartesian closed categories have pullbacks and pushouts [40]. Figure 2 combines them in a pullback-pushout diagram sometimes known as a Dolittle diagram from the Push me-Pull you creature invented by Hugh Lofting in his book *The Story of Dr Dolittle*. (It is by the way the formal mathematical representation of another mythical creature the Schrödinger cat). Y as a subcategory of C is pulled back over X in C. Alternatively and co-currently and co-terminously the projection Y is pushed out of the left entangled state over the projection X. In set theory the right gives the joins $X \cup Y$ and the left the meets $X \cap Y$ but as category theory is more precise $X \cup Y$ only refers to the disjoint union X, Y. Other kinds of conjunction have to be specifically defined. The whole diagram in Figure 2 is a universal logic gate for X, Y.

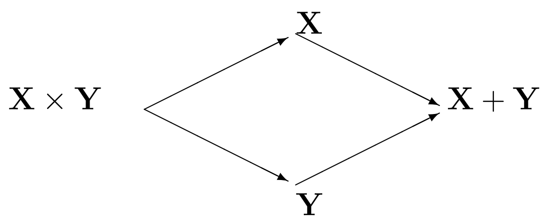

Fig. 2: Dolittle Diagram of Pullback/Pushout: **Y** of **X**

It is possible in this way to handle context so that the limit $X \times Y$ obtained by pulling back Y over X may be restricted to a particular context c (a subobject which may be an object or subcategory) of C (Figure 3). Examples of this can be found in various types of information systems [23] like law, expert systems [47], object-relational databases [48] and consciousness [22, 17].

The cartesian category on the left of Figure 3 is a left-exact category of limits where all the concepts in the left class of Table 1 (in Part I of this paper) reside. It was sometime known as a **LEX** (Freyd & Scedrov [20] section 1.43; Taylor [52]

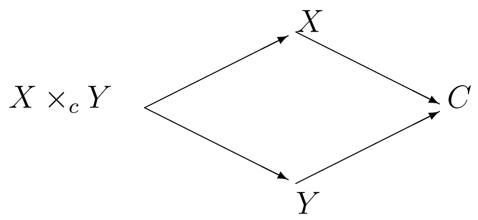

Fig. 3: Diagram of Pullback of Y over X in the context of c, a subobject of C

at footnote 2 p259). The category on the right (**C** in Figure 3) is the right-exact of co-limits where all the concepts of the right class of Table 1 (in Part I of this paper) are to be found. This is the classical world of the basic components of the Universe, the elemental particles, atom, molecules, physical structures, classical objects we recognise including the mind of the observer. These are all local structures that is subcategories of the category **C**. Thus for instance the concept of sets exists in the mind. Co-units in the subcategory of sets form a disjoint union but in general do not need to be discrete.

It might be remembered that the original paper on the EPR paradox [18] is entitled *Can Quantum-Mechanical Description of Physical Reality be Considered Complete?* On the first page Einstein and his co-authors are describing in effect left- and right-exactness: "if, without in any way disturbing the system, we can predict with certainty the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity" (at p777). Physical quantity is on the left and physical reality on the right.

The nature of the Universe is that there are innumerable paths between any two objects but only one effective resultant for each pair. This is the structure of partial order where there is only one arrow at most between each pair of objects. The importance of partial orders was originally recognised (for sets) by C S Peirce [44]. Again it is to be noted that the logic of a partial order is Heyting.

The category **C** is a signature of sorts Σ, that is the categories of observables. Taylor [53, 54] uses for Σ the term *half-bits* which is reminiscent of Reichenbach's three-level logic [45] with truth value of *half*. Taylor however is interested in foundations of pure mathematics where **C** can be a concrete category with Σ as a set. In QIS we are concerned with physics where the set does not exist and Σ is a large category. Likewise the pre-order Σ^{Σ} corresponds to Taylor's poset and his fundamental condition for monadicity [53]. The modern mathematical concept of the monad (*triple* in Barr & Wells [6] at 14.3, and [5]) as abstract adjunction viewed as an endofunctor seems to correspond to Leibniz's use of the term monad [36], for reflective subcategories are monadic and idempotent (Taylor at example 7.5.10(a) [52]).

The left side is the quantum world. It has a non-local entangled structure as a pre-order. The pre-order is a partial order without the anti-symmetric condition of isomorphism between a and b whenever $a \leq b$ and $b \leq a$. Removing this condition removes the restriction to locality. The nature of the pushout is any partial order on the right is one of the equivalent quotient class of the pre-order on the left. This captures the essence of the collapse of the wave function or Everett's multi-world interpretation [19]. If X and Y in Figure 2 represent the double slits in the experiment of that name the entangled quantum state on the left give rise to a diffraction fringe pattern on the right.

Consider the conjugate variables P and Q. P (the classical momentum) is matter in motion and is to be found in C. This is Bohr's individuality postulate. Q is a generalised coordinate of space which need not exist *a priori* but is generated by P. That is Q is pulled back over P to form the limit $P \times Q$ as in Figure 4(a) which describes motion in space (Bohr's quantum postulate). Heisenberg's uncertainty principle tells us that the minimum limit is a pullback over the Planck's constant $\hbar$ which is an energy object (a special case of the context c of Figure 3 above) and a co-equalizer in C. The Heisenberg uncertainty principle is represented by Figure 4(b).

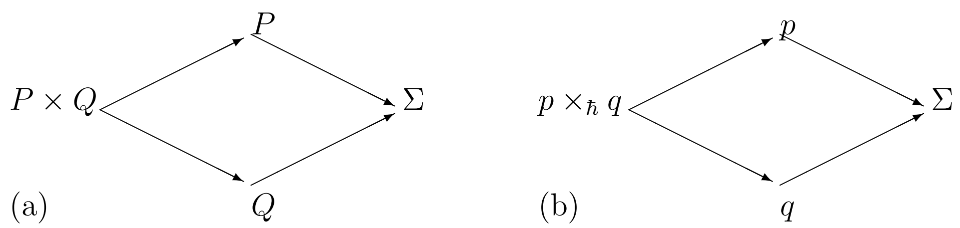

Fig. 4: Pullback of (a) Q over P; (b) uncertainty q over the uncertainty p (Heisenberg's principle)

For quantum computing in qubit mode we have Figure 5(a). ψ is the entangled state of the qubits $|0>$ and $|1>$. In hobit mode there are only the non-local quantum bits corresponding to the initial object $\perp$ of the cartesian closed category and its terminal object $\top$ giving diagram Figure 5(b). On the right is the monoidal category of the Universe as a whole with the conventional * symbol.

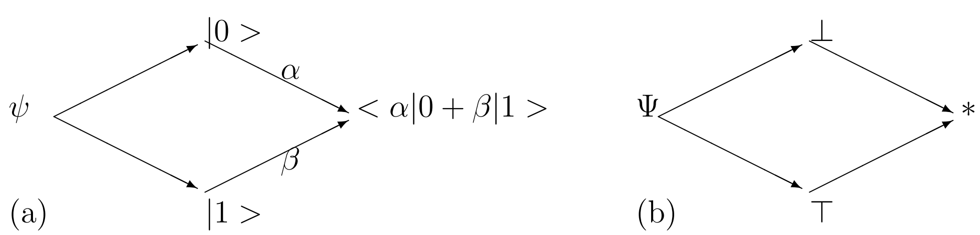

Fig. 5: Diagram of Pullback of (a) $|1>$ over $|0>$; (b) $\top$ over $\perp$

A fuller system of arrows is given in Figure 6 from the perspective of the adjoint relationships in the pullback (including the apparent structure of entanglement) and illustrates many of the corresponding concepts in the third column of Table 2 (in Part I of this paper). The fundamental $\Sigma \dashv \Delta \dashv \Pi$ arises from the pullback view of the Abstract Stone Duality [32, 52, 33] between pullback and pushout. Bohr's individuality of elementary process is Σ. Bohr's postulate of the interaction of object and instrument is given by the adjunction $\Delta \dashv \Pi$. The diagram therefore not only gives a formal specification for the three postulates of Bohr but goes further to give the adjointness of the observer.

observable $\dashv$ observer $\dashv$ observation

The observer is right adjoint to the observable and left adjoint to the observation.

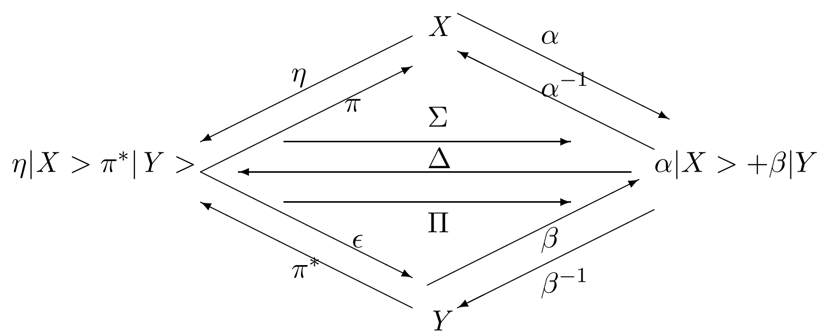

Fig. 6: Pullback of Y over X

Therefore as a universal diagram the pullback-pushout adheres to the correspondence principle and includes both classical and quantum computing. The categorial version not only accords with the quantum scene but also in the classical realm in connecting existence with the observer in the sense of Descartes' *cogito ergo sum*, with general empirical philosophy and even with the speculations of Bishop Barclay.

The universality of the diagram includes also classical cases and collapses the observable-observer observation onto the two-level ontological/epistemological relationship [21]. The pullback also captures the various aspects of complementarity: correspondence between left and right exactness; the wave particle duality in the arrow/object identity; and the canonical conjugate categories $\mathbf{P}, \mathbf{Q}$.

3 Results and Conclusions

It is apparent that the prospect of building a quantum computer forces us back to re-evaluate the fundamentals of quantum theory from a constructivist perspective.

This requires a fresh look at the Copenhagen and other interpretations. As Nadeau & Kafatos note 'most physical scientists have tended to relegate Bohr's views to a file drawer called philosophy .... we must open that drawer and review its contents' ([41] p39).

Quantum mechanics itself tells us that its own subject matter is non-local. By the application of the concepts of anticipation and realisation from an alternative object viewpoint based on Rosen's theory of anticipatory systems, we can see that we need a non-local form of language description where set theory and the axiomatic approach have limitations. Category theory can be used in a non-local mode of formal description as strong anticipation. Unfortunately category theory as a culmination of algebra topology and geometry was not advanced sufficiently at the time to be utilised by the founders of fundamental quantum theory or for that matter by Einstein. So we shall never know how quantum theory and the theories of relativity would have been advanced and perhaps merged with a universal formal tool in their hands. In particular the arrow is a language of interaction not of a bound background. Just as the twentieth century freed these theories from the fixed frame of the ether. So the twenty-first century is able to escape the mathematical ether of a set-theoretic co-ordinate reference frame.

We can see from this cursory glance the relevance of fundamental ideas like limits and adjointness which have only really been appreciated since the advent of category theory. By a comparison of existing theory with that of a possible categorial representation, we can already glimpse a deeper understanding. It is the 'third way': the natural inherent intuitionistic logic and non-local constructive approach for quantum information systems.

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