Formal Representation of *Process & Reality*: Whitehead's relational theory of space with Category Theory

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Abstract

Category theory was not sufficiently developed in his lifetime for Whitehead to apply to his speculative metaphysics of *Process & Reality*. The natural (assumption-free) topos as a cartesian closed category is now able to conceptualise formally the inherent space-time structure of Whitehead's extensional space that he appreciated is beyond the metrics of the classical mathematics he helped to develop. This paper explores that structure to confirm Whitehead's underlying belief that formal metaphysics needs to be founded in physics rather than finitary mathematics.

Keywords: Extensional space, process, formal metaphysics, natural topos

1 Background

Alfred North Whitehead was one of few to appreciate 'the theory of linear extension'[1] (Grassmann 1844) of the eminent philologist Hermann Grassmann (1809-1877) that was to lead to linear algebra, vector spaces, differential geometry and the mathematics that underpinned much of 20th century science ((Penrose 2004) at pages 203sq). Grassmann's studies contained a germ of category theory to be pursued here for in the meantime it has led to mathematical topics like universal algebra, topology and homotopy all of which are subsumed in category theory. In particular Grassmann's insight[2] allows geometry to escape from the metric dimensions of Euclid. That escape epitomises the natural ('assumption-free') essence of the *topos* in metaphysical category theory, the subject of this paper. That now well describes the connectedness that Whitehead outlines for the structure of his 'cosmology' but which he was unable to represent formally even as a world-class mathematician of his age.

Whitehead himself pursued Grassmann's ideas with his own *Universal Algebra* (Whitehead 1898) but his early career may be characterized as a somewhat frustrated author of mathematical texts. *Universal Algebra* led to his election as a fellow of the Royal Society of London but disappointed him that the work was not generally understood. His projected second volume was therefore abandoned in favour of a joint treatise with his

[1] *Die Lineale Ausdehnungslehre, ein neuer Zweig der Mathematik*

[2] An insight that may very well have come from Grassmann's studies in natural language.

student Bertrand Russell on the logical basis of mathematics. He and Russell attended the renowned Paris 1900 International Congress of Mathematicians in Paris and felt further inspired by interaction with the likes of Hilbert, Frege and Peano. The outcome was the well known but little read *Principia Mathematica*.

The work was probably as frustrating to write as it is to read. Part II (at pp 328-383) of the first volume attempts to define the cardinal numbers 1 and 2 but without success. Volume II on the other hand devotes 724 pages in an unsuccessful attempt to formalize the arithmetic axioms of Peano and fails even to establish the fundamental $1+1=2$. The explanation for all this we now know with hindsight is because the natural topos lacks a natural number object[3]. Whitehead's dismay and disappointment with the whole project of the *Principia Mathematica* is understandable. Not surprisingly the fourth volume on geometry was never published even though apparently much of it had been written. Indeed Whitehead did not involve himself with the second edition of 1925-1927 at all. However it appears that in the meantime Whitehead was diverting to a physics approach to geometry that emerges in *Process & Reality*(Whitehead 1929) The fourth volume was therefore turning out therefore to be a nightmare and in some conflicting transitional state. For Russell alludes to this in the preface of (Russell 1914) when commenting on the problem of scientific method in philosophy:

> I have been made aware of the importance of this problem by my friend and collaborator Dr. Whitehead, to whom are due almost all the differences between the views advocated here and those suggested in *The Problems of Philosophy.* I owe to him the definition of points, the suggestion for the treatment of instants and "things", and the whole conception of the world of physics as a construction rather than an inference. What is said on these topics here is, in fact, a rough preliminary account of the more precise results which he is giving in the fourth volume of our *Principia Mathematica.*((Russell 1914) preface p 8).

We are concerned here with the subject matter of the fourth volume so far as it relates to Whitehead's concept of extensional space but not from the viewpoint of the history and sociology of science. Readers interested in that perspective are referred to the studies of Patrick J Hurley [4] who cites Whitehead's displeasure at Russell's disclosure of the fourth volume material in (Russell 1914) expressed in the letter to Russell:

> I am awfully sorry, but you do not seem to appreciate my point. I don't want my ideas propagated at present either under my name or anybody else's – that is to say, as far as they are at present on paper. The result will be an incomplete misleading exposition which will inevitably queer the pitch for the final exposition when I want to put it out. My ideas and methods grow in a different way to yours and the period of incubation is long and the result attains its intelligible form in the final stage, – I do not want you to have my notes which in chapters are lucid, to precipitate them into what I should consider as a series of half truths . . . ((Russell 1968) at p.78).

[3]The natural number object can only be concocted by assuming some successor function and that requires an axiom of choice to import some closed world assumption.

[4]See www.religion-online.org/showarticle.asp?title=2469

Whitehead's other writings of the ensuing 'period of incubation'(Whitehead 1906; Whitehead 1914; Whitehead 1919; Whitehead 1920; Whitehead 1922; Whitehead 1926; Whitehead 1933; Whitehead 1934) suggest that Whitehead was for some time confident that his extensional theory of space presented in Paris in 1914 (Russell 1914)) [5] and consisting of material intended for the fourth volume could still be expressed mathematically as distinct from Russell's 'half truths'. Whitehead's alternative 1922 theory of relativity (Whitehead 1922) probably marks the watershed after which he realized that it is a relativistic quantum world we inhabit beyond classical mathematics. A lesser mathematician might have persevered with the tensor mathematics but he clearly appreciated that his earlier logicalism was inadequate to articulate his speculative metaphysics. Indeed Gödel confirmed that in his doctoral thesis (Gödel 1929) (at p2) by expressly using the axioms of Whitehead's *Principia Mathematica* as the basis to prove that first order predicate logic is complete but only for closed systems. Metaphysics on the other hand is of higher order but amenable to the intuitionistic internal language of the natural topos derived from physics and therefore outside of Gödel's theorems.

Of course this strand from Grassmann was only one of many influences on Whitehead as comes through from the text of *Process & Reality*: it includes Plato, Aristotle, Leibniz, Newton, Kant, Locke, Hume, etc; but beside the general proposition that science progresses better when supported by mathematics there seems little from them of direct relevance to formalizing the geometry of space. There were also contemporaries like Poincaré, James, Bergson, Dewey, Husserl, Einstein, Bohr, etc, who get little or no mention in *Process & Reality* but who nevertheless were providing a climate of thought operating heavily on Whitehead's mind that it was necessary to escape the limitations of classical mathematics – but again these seem of little direct relevance here. What at first sight does appear more relevant, but in the foreground rather than background, is the content of Part IV of *Process & Reality* itself and the work of those who have since sought to build on it.

2 Foreground

The extraneous evidence outlined above suggests Part IV of *Process & Reality* entitled simply *The Theory of Extension* is the material written much earlier as proposed contents for the projected fourth volume of *Principia Mathematica* on geometry that was never published. Certainly it appears as an insert differing markedly from the rest of *Process & Reality* as the only part in any way mathematical. It is geometric in tone but in a very idiosyncratic style reminiscent more of Venn diagrams[6] than Euclidean geometry. On the one hand it does not adhere to the strict logical principles adopted in the first volume of *Principia Mathematica*. It makes assumptions that are beyond self-evident primitives and lists *a priori* definitions that are more than mere labels, as banned in the introduction of the first volume. On the other hand it is clear from internal evidence of the text that it is much more than an opportunity to get published material already written for another occasion. It incorporates more recent published work of others. For example definitions

[5] There is apparently no original English version extant but details of its publication in French with its subsequent translations into English and commentary may be found at the religion-online website given in the footnote above.

[6] in a couple of pages ((Whitehead 1978) pages 295sqq.)

of Professor T de Laguna (Whitehead 1978) (at pp 287, 295 & 297) are fundamental to the main thrust of Part 4 and indicate that the whole subject of the extensional theory of space had been recast in Whitehead's mind. This is also further evidence (as from his letter to Russell cited above) that he was struggling perhaps for nearly twenty years with a formal description for space. Part IV is the then current version of Whitehead still trying his hand at representing connectivity in the reality extension of his world of process.

Some of his observations are very pertinent here. Thus the overlap in the diagrams he makes into 'ovate classes'. This is a perspective of universal limits in category theory the significance of which was not appreciated until the 1970's. The impossibility of producing adequate diagrams[7] to represent such features also add weight to the proposition that we need to turn from mathematics to physics for nature produces an abundance of limits and co-limits, indeed everywhere all the time. However the mathematics of Part IV does not really go anywhere and Whitehead did not take it any further in the remaining twenty years of his life.

However the disciples of genius often with great enthusiasm attempt to take the work way beyond where the master would have gone and subsequent events show that the subject of Whitehead's connectivity has many potential onward paths to pursue. Geometrical connectivity is one aspect of atomicity and raises issues of whole-part relationships. Whitehead has in this context spawned interest in some new disciplines like holism, point free geometry, mereology, and mereotopology. These have generated a considerable literature[8] and attempts to define new formal systems of logic. Examples of these are the work of (Clarke 1981), (Simons 1987) and (Casati & Varzi 1999) but the latter have demonstrated that the formal representation can be reduced to Boolean systems. Boolean logic however is not inherently constructive and does not have the required intuitionistic logic required by physics. Although the subject matter of that field of work is within the ambit of this paper they will not therefore be examined in detail here as the end result is a 'null return'. These are now mainly of only historical interest.

Although Einstein's relativity and quantum theory were contemporary with his 'period of incubation' and a clear catalyst for *Process & Relativity* Whitehead made no serious attempt to include any quantum mechanics in his theories. Perhaps his brief abortive incursion into the subject of relativity (Whitehead 1922) dissuaded him. This century Michael Epperson has made an 'attempted correlation of quantum mechanics and Whitehead's cosmological scheme' (Epperson 2004). That attempt has taken the form of a painstaking recasting of Whitehead's metaphysical categories in a Hilbert space with Dirac notation. This unfortunately seems rather to miss the point that a Hilbert space is composed of points which are just numbers, even though rather sophisticated numbers, but are ghosts from Whitehead's discarded former life of *Principia Mathematica.*

More recently Epperson has published *A Topological Approach to Quantum Mechanics and the Philosophy of Nature*(Epperson 2013). 'Topology' refers to the use of a sheaf theory approach from co-homology. The publishers had issued pre-publicity with the title *Foundations of Relational Realism: Quantum Mechanics, Category Theory, and the Philosophy of Alfred North Whitehead* The substantial change from 'category theory' to

[7]The diagrams of transition functions between overlap of manifold patches in twistor cohomology, for example fig 33.17 in (Penrose 2004) at p 988, are perhaps more advanced developments of ovate classes in Whitehead's diagrams

[8]and vocabulary like *gunk* for any whole with proper parts.

‘topological approach’ perhaps suggests that some original aim to use category theory was not realised. The change is quite significant because it undermines the decoherence thrust of the book. Metaphysical category theory can track the space-time development of the quantum wave function. Application of topology immediately collapses the wavefunction *a priori*. Sheaf theory is a finitary description of the *pullback* in full category theory. We have already shown in 2002 how this provides a simpler yet more sophisticated approach to quantum theory (Heather & Rossiter 2002a; Heather & Rossiter 2002b). The significance of quantum decoherence within process is more simply represented as monadic composition in the natural topos.

3 the significance of category theory

Around the time of Whitehead’s death in the 1940’s formal ‘category theory’ emerged to subsume algebra, geometry and topology as a formal metaphysical language that is now able to integrate his natural philosophy and mathematics to culminate in what we might explore here as the implied formal ingredients of *Process & Reality* as it climbs up through two levels from models to metaphysics. A model reduces reality that metaphysics generalizes.

Just as a mathematical theory is an instantiation of the world so the world is an instantiation of metaphysics. For historical reasons category theory has had to develop from within classical mathematics and current text books deal mainly with the category of sets that resides within Whitehead’s discarded mathematics of his early period and therefore cannot deal adequately with his speculative metaphysics. For as metaphysics generalizes the dynamics of nature, metaphysical language relates to natural process without the need for the arbitrary axioms of mathematics. Fortunately therefore metaphysical category theory is simpler than the category theory of classical mathematics and also greatly simplifies the natural language descriptions that flowed from the pen of the author of *Process & Reality* that are difficult for those of us not endowed with the power of his mind. The formal categories are therefore simpler than the natural language expressions but it is a simplification satisfying his own observation that “the only simplicity to be trusted is the simplicity to be found on the far side of complexity.”

Rather paradoxically mainstream science a century later is still trying to understand our world using the models based on the concepts of his early mathematical period rather than the informal categorical approach enunciated in the 1929 *Process & Reality* of his later philosophical period. The current mainstream position at the turn of the twenty-first century is probably well summed-up in Penrose’s encyclopaedic tome entitled *THE ROAD TO REALITY, A Complete Guide to the Laws of the Universe*:

> There have also been other intriguing radical proposals, such as those of Richard Jozsa and of Christopher Isham which employ topos theory. This is a kind of set theory arising from the formalization of ‘intuitionistic logic’ (see Note 2.6), according to which the validity of the method of ‘proof by contradiction’ (§2.6, §3.1) is denied! I shall not discuss any of these schemes here, and the interested reader is referred to the literature. Another idea that may someday find an significant role to play in physical theory is category theory and its generalization to n-category theory. The theory of categories,

> introduced in 1945 by Samuel Eilenberg and Saunders Mac Lane, is an extremely general algebraic formalism (or framework) based on very primitive (but confusing) abstract notions, originally stimulated by ideas of algebraic topology. (Its procedures are often colloquially referred to as 'abstract nonsense'.) ((Penrose 2004) at p.960)[9].

The typo 'an significant role' in this short extract suggests that Sir Roger was unhappy with this sentence and had not finished editing it. The reference to Jozsa is to an unpublished thesis he supervised and those to Isham relate to books he edited but none seem to give any adequate treatment of a topos or category theory. Although the book has in its title 'A complete guide to the Laws of the Universe' nevertheless on its own admission is confused by the notions of category theory as 'a kind of set theory'. The 'generalization to 'n-category theory' relies on number and is therefore limited to finitary mathematics and its restrictions. The extract clearly discloses a serious misconception on the significance of intuitionistic logic in constructive mathematics. The single exclamation mark about the validity of proof by contradiction raises a shadow over the whole thousand pages of the book and fuels our belief that any scientific theory today is suspect unless it can be validated by category theory. It follows that such validation also tests the correlation of any scientific theory within Whitehead's scheme of speculative metaphysics. It was probably Alexandre Grothendieck of the Bourbaki group in France who was the first to see the depth (or more accurately the 'heights') of significance in the topos. Aristotle was of course responsible for promoting the metaphorical connotations of the simple word for *place* in Classical Greek and there is a parallel abstract usage to be found in literary contexts. A major feature in Aristotle that cannot be captured by finitary mathematics is the macrocosm-microcosm relationship where the part has the characteristics of the whole. Whitehead alludes to this relationship in *Process & Reality* and seems to use the terms interchangeably with 'macroscopic and microscopic'. Finitary mathematics is unable to represent the relationship directly because a set cannot be a member of itself in axiomatic set theory and cannot be proved to be consistently defined in naive set theory. Likewise unfortunately the Grothendieck topos does not manage to escape from its Bourbakian roots in Hilbert's finitary methods[10] which also serves to make it unnecessarily complicated. The same over-complexity may be found in the standard category theory texts of classical mathematics[11]. The approach from physics on the other hand by identifying the arrow of category theory with process in nature greatly simplifies the complexity and enables category theory to act as an Occam's razor and as a very powerful scientific tool. For a parallel view as an alternative natural philosophy, see (Heather & Rossiter 2011) where two texts that have proved seminal are examined to show the limitations in the use of the category of sets to represent the internal structure of the topos as a cartesian closed category – which corresponds to Whitehead's 'extensional space'.

[9]the Note and § numbers refer to the Penrose' book.

[10]Colin McLarty recently claims that 'the entire Grothendieck apparatus' is of weaker strength than finite order arithmetic (McLarty 2013).

[11]No attempt will therefore be made here to give a comprehensive list of these texts as they are not directly relevant.

4 The Topos: Archetype of Natural World

The archetype of the natural world is the topos, in its early days formally defined as a Cartesian closed category with subobject classifiers and informally as a generalised set. Johnstone in his preface to (Johnstone 2002) lists thirteen alternative descriptions that have been applied to the topos (pp. viii &sq). Many of them like for instance "A topos is a generalised space" still carry hangovers from sets. We would recommend as an informal definition: "The category of categories of categories". To some this may only confirm categories as "abstract nonsense" but it is accurate and makes the recursion explicit. The topos sums up all that we have said in this paper. It is the ultimate intension existing as an identity natural transformation in any extension given by the internal categories, subject to the locally cartesian closed condition with the preorder structure and an intuitionistic logic that is Heyting and which is more general than Boolean. There is a unique arrow from the source of the World to every object in it and a unique limiting arrow directly between any pair of objects as well as a repletion of indirect co-limiting arrows between them. These relationships satisfy our empirical perception of 'the laws of physics'.

To satisfy its holistic nature the World must emerge top-down. That is to say no more than that if the Big Bang happened it potentially contained everything that ever existed. However it is easier to explain bottom-up by treating the role of the arrow as a natural expression of process with an identity arrow as intension and a distinguishable valued arrow for extension. Nevertheless while in naive category theory the simplest identity arrow may be treated as an object, it is convenient to begin with a category of three composing objects as a generalisation of any possible category. This is shown in Figure 1 with the next higher identity arrow (the functor) composing extensional arrows between objects. The next higher identity arrow is the locally cartesian closed natural transformation composing categories with ordinary functors as extensional arrows between categories as also shown in Figure 1. The highest level arrow is also a natural transformation which composes structures of categories and functors. It is this identity natural transformation that constitutes the full cartesian closed category of a topos as in Figure 1. However, the natural arrow is double-headed to represent as a composition of the adjoint functors but with a parity arising out of the order of the adjointness. Although the manner just explained bottom-up may be easier to understand the diagram because of the way that models are usually built-up, nevertheless process can only exist as a whole which requires it to devolve top-down. The circles with an arrow head represent an identity arrow. The particles in the 'standard model' of classical physics would be represented to first order by the smallest identity arrows. However in process physics a particle can't exist in isolation and the minimal identity object is the triangle in the diagram. Each such triangle represents a natural occasion or "actual event" as first introduced by Whitehead (Whitehead 1929).

The whole is just a recursive system with closure at four levels consisting of three open interfaces. Figure 1 shows the three interfaces for composing arrows (ordinary, functor, natural transformation) with the four levels (identity arrow, identity functor/category, higher-order identity functor/category, identity natural transformation/topos). The 'higher order is an intrinsic component of the topos structure and described as the property of being 'locally cartesian closed'. Thus from the top-down perspective of the topos it would be more appropriate to designate the 'higher-order as lower order. The locally closed cartesian closed categories are at the level of number and metric space-time. With an ar-

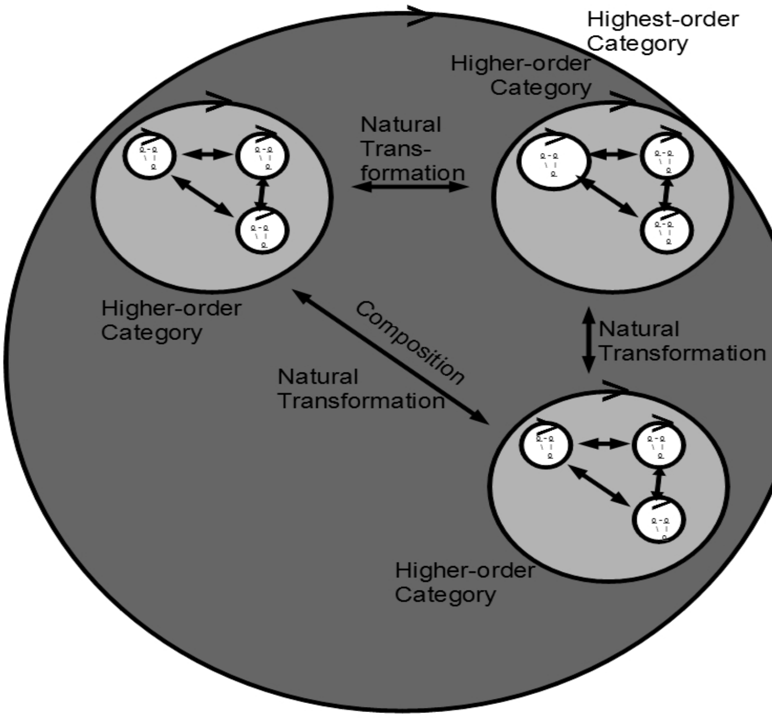

Figure 1: Natural Transformations of Composing Functors themselves compose in the highest possible category, a Topos

bitrary initial object or 'bottom' they are therefore Boolean and explain the approximate success of Euclidean models in physics. A prime example are Einstein's theories of relativity. These bridge the gap between the measurement of physics and the mathematics of relativity.

The diagram exhibits the natural recursive nature of the structure. It also demonstrates connectivity from any object to any other object. It is possible therefore to get from any object to any other object directly: or indirectly with possible local variations through any other internal path. This is a natural structure because it is obtained by simple induction applied to the notion of process without any assumptions. It is also 'natural' as formally defined in category-theoretic terms.

5 Future Directions

The next step will be to represent formally the entity types of *process* that populate his *reality* but there is no space to attempt that here. Only a few 'simple' concepts are needed: the World is a topos with monadic objects related by contravariant functors with natural transformations as units of adjunction. These are sufficient to identify formally the Whiteheadian vocabulary encompassing the likes of the ontological principle, actual entities and occasions, eternal objects, congrescence, creative and emotive advance of becoming, public and private, prehensions, nexus, primordial nature, emergence, etc, together with their other postmodern counterparts. Nevertheless it will be a long road to represent Whitehead's *Process & Reality* formally but until it can be studied in this way his speculative metaphysics can never be fully understood nor expect the impact it deserves.

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