Anticipation in Communication: The Example of the Cartesian Monad in Music

Nick Rossiter & Michael Heather
Computing Science and Digital Technologies
Northumbria University
Newcastle NE1 8ST, UK
nick.rossiter1@btinternet.com; michael.heather@trinity.cantab.net
http://www.nickrossiter.org.uk/process/

Abstract

Music is a testing challenge for formal information systems even using the power of the Cartesian monad for universal design. The concept of the monad developed from the term originally introduced by Leibniz needs to have the capability to present a musical performance as a categorical composition over time signatures that proceed in successive adjoint steps: overall the monad looks back and its comonad looks forward. The physical complexity of each musical note operates in its respective time-frame as a co-limit. The monad as process handles dynamic aspects in the Cartesian closed category of the *Topos* as it follows the variation of data for players and score together with their inter-relationships. The monad and its dual the comonad provide a natural basis for anticipation.

1. Introduction

Much work has been done on computer representations of music at the physical level. Developments such as K-nets (Klumpenhouwer, 1991) provide a way for representing transformations from one pitch-class to another. A pitch class is all notes an octave apart, for example all C available on a piano. In a classical system there are 12 pitch classes, one for each note on the 12-note scale. K-nets represented a fundamental change from a set-theoretical approach to music theory into a transformational one. Earlier the emphasis had been on the pitch classes being treated as sets of elements, each element being a note within the item. This enabled chords and other combinations of notes to be modelled. The transformational approach extended this technique by adding a transition from one pitch class to another to capture the dynamic possibilities within a musical piece. Such a transition from, say, K_1 to K_2 was tightly structured, with the target K_2 being isographic to a source K_1. Isographism is similar to isomorphism but within a graphical context. The cardinalities of the source and target nodes must be the same. K-nets are therefore a very disciplined but restricted way of moving from one pitch class to another: they can handle the situation where the labelling of nodes is consistent from one system to another and where the transformations are classical as, for example, within the 48 preludes and fugues (Well-Tempered Clavier) of JC Bach, handling all 24 keys.

A more general form of K-nets was defined (Lewin, 2002), an extension of Klumpenhouwer's work in an attempt to make the graphs and their transformations more general. These have been termed L-nets by O'Donnell. L-nets extend a node from being a static collection of pitch classes to a network of transpositions, giving a transformational model, allowing numerous graphic possibilities for representing a single pitch-class set.

2. Previous Attempts with Category Theory

L-nets still have their origin in set-based graph theory. It was not long before the potential was seen for a move to category theory with the nodes in the graph becoming categories and the edges becoming functors. Category theory should facilitate the development of a logical approach to music, which can be mapped into one of the physical approaches for implementation. Towards this aim a massive tome *The Topos of Music*, 1335pp long, was produced (Mazzola, 2002), bringing together many of the recent advances in the theory of music. The title is, however, misleading with a formal topos approach, based on the Cartesian closed category, not attempted. In the preface (p.v) it is stated that the word topos is used in the style of Aristotle's or Kant's topic. Chapter 19 *Topoi of Music* gives an overview of the Grothendieck Topology but does not relate the topology to music. In later sections the word *topology* is frequently used but is nowhere elevated formally to a topos. Section XVI, containing Appendices C-I, deals with many categorial concepts but not in a musical context. Because the book is disjoint in its treatment of the topos and music, it has failed to achieve its aim as highlighted in the title. The most relevant section for the application of category theory to music is Chapter 6 *Denotators,* a concept developed further in collaboration with Andreatta.

The subsequent paper (Mazzola and Andreatta, 2006) develops the idea of a category of directed graphs with objects as notes or chords and edges as musical operations such as transposition. The formalism of K-nets in category theory as denotators is developed in detail as a digraph, with vertices and arrows. In music the vertices are pitch classes and the arrows are operations; between any two vertices, there may be multiple arrows and an arrow may map from a vertex to the same node, a loop. A path in a digraph is a sequence p = a_1, a_2, ... a_i where i is the number of arrows (a) in the digraph. The operations are the elements of a group T/I (translation/inversion), that is a bijective mapping ensuring that p is invertible. In category theory T/I is a category with one object Z_{12} (the 12-note scale) and automorphisms f: $Z_{12} \rightarrow Z_{12}$ The authors acknowledge that Z_{12} is far too restrictive from the articulation viewpoint, replacing it by a four-dimensional real vector space R_4, where the coordinates represent onset o, pitch p, loudness l, and duration d, in an appropriate parametrization by real numbers. Their use of the powerobject for collections of notes as a basic type enables chords to be represented, the powerobject being any combination of notes, permitted from Z_{12}/R_4. The complex categorial formalism ultimately developed involves limits, co-limits, presheaves, powerobjects, Yoneda. The references to the categorial literature are very general but it appears that their approach owes much to the uncited Eilenberg-Moore category: the pullback of the category of presheaves on the Kleisli category along the Yoneda embedding.

More recent work developed the generalised Poly-K-net or PK-net (Popoff et al, 2015). PK-nets enable heterogeneous collections of musical objects to be naturally compared and manipulated (Popoff et al, 2016a). In particular the cardinalities of the source and target nodes do not have to be the same and the labelling of the nodes in two different approaches may be varied to suit the genre. Five main categories are developed, one PKN_R for the underlying PK-net and four others as homographies of the PK-net. Four functors are defined, relating the categories. Natural transformations are used to generalise isographies. Their work does not employ explicitly Cartesian closure so does not appear to be from a topos viewpoint. The dynamic aspects involve a combination of functors and natural transformations, following a Godement calculus approach.

Problems occurred with the sets representing the graphs, resulting in their replacement by the category of relations REL (Popoff et al, 2016b). This facilitates handling relationships but is inferior to the pullback, which is locally Cartesian closed and hence adaptable to a topos view.

The main findings from the literature review are that the approaches do not provide a natural correspondence with music. In particular the conversion of the K-nets and successors to categories, functors and natural transformations is categorification at a low-level of the set theoretic graphical structures, on a 1:1 basis. However the denotators approach (Mazzola and Andreatta, 2006) with the apparent use of the Eilenberg-Moore category comes closest to our approach presented here and the PK-nets or denotators could be very useful as a basic representation of the notation in the score.

3. Natural Category Theory

The alternative approach to categorification is to search for a natural correspondence between music and category theory. Music is a composition of sounds from point to point as a succession of transitions. Category theory also involves, as a central tenet, the principle of composition, from the target object of one arrow to the source object of another. In both music and category theory the arrows have a direction from a starting point to a closing point, though loops may exist. An isolated point in music is a sound without context while an isolated object in category theory is simply a set. It is the processes that map from object to object that provide the naturality.

The practice by a performer of playing a score is the personal communication, often highly intensive, of the piece to a listener. When performing a player is at the same time both looking forward to what is to be played next and looking back at what has just been played. The process of music is indeed similar to that for transactions in a database system, where monads have been used to represent process (Rossiter et al, 2018). There are however some significant differences. Aestheticism is an important part of music, covering aspects such as style and improvisation, subject to the rules of intonation. So while in database systems it would be a major deficiency if transactions were not always perfect to the letter of the requirements, in music variation through expression is an integral part of a performance, involving a departure from the score in aspects such as phrasing, rubato and articulation. It is necessary to move from the syntactic level of Shannon's communication theory to the semantic/aesthetic level.

The starting point for a data structure suitable for music is the locally Cartesian closed category, based on relationships within a product (pullbacks, limits), connectivity (exponentials), internal logic (λ-calculus), identity (from the limit), interchangeability of levels (object to category-object), hyperdoctrine (adjointness between quantifiers and the diagonal). Ideally this should be extended to a topos, the data structure of choice in applied category theory. This requires the definition of relationships within a coproduct (co-limits), an internal intuitionistic logic (Heyting), a subobject classifier (query) and a reflective subtopos viewpoint (query closure).

In music the structure of the nodes is potentially very complex and diverse ranging from a single note through tonal chords and dissonant combinations to microtones. On the piano powersets of integers may suffice but the Cartesian space will be more complex for the violin. Cartesian spaces as pullbacks can be constructed for real numbers through smooth manifolds expressed as differential forms. Another dimension is the articulation, described earlier.

The intension/extension relationship plays a central role in music. The intension is the type; the extension is the collection of instances that satisfy the type. It is not as simple though as a hierarchy of types. There remains a philosophical dimension to the design. The Universe contains everything. The Universe of Discourse (UoD) is that section of the Universe of interest to our application. By the laws of physics we cannot isolate any part of the Universe but we can identify a section for our work. In this case the intension is the Universe and the extension, UoD, is the world of music. A musical manuscript is extensional to the UoD of Music as one of the objects in this universe but intensional to the manuscript's variants and their performances. Variants include changes to the score (composer initiated or developments after composer's death), rehearsal (conductor initiated) and performance. No two performances are ever the same. In the next two sections, we bring together these ideas in the formal definitions of the topos for a data structure and of the monad as a process, operating inside the topos.

4. The Topos as the Data Structure

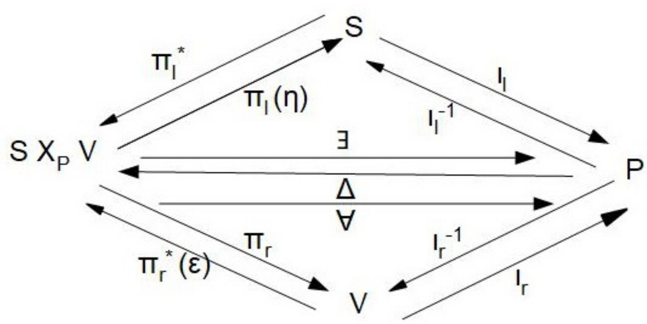

Figure 1. Locally Cartesian Closed Category. Relationship $S X_P V$ of Score by Variant in Context of Performance. Categories: S score, V variant, P performance.

We now look at an example of a locally Cartesian closed category in Figure 1. This represents the pullback relationship $S X_P V$ of Score (S) by Variant (V) in the context of Performance (P). As a hyperdoctrine, there is adjointness between the functors: the existential quantifier is left adjoint to the diagonal Δ, which is right adjoint to the universal quantifier. The arrows Π and ι are projection and insertion arrows respectively with the subscripts l and r indicating left and right. The diagram appears to be intensional but the extension is held within each category-object as a

Dolittle diagram, a pullback that is also a pushout when Π_l is monic (1:1). So each node in the diagram is an internal pullback-pushout square, with limits and co-limits, relating the definition to the instances. Dolittle diagrams are also known as adhesive categories, which are readily embedded into a topos, yielding our objective of a topos as the data structure. The extension for the score will contain the notes, perhaps as the digraphs or denotators described earlier. The full data structure will be more complex with further locally Cartesian closed categories representing other entities such as composers, conductors, musicians and venues. These can be pasted together to give complex relationships (Rossiter et al, 2018).

5. The Monad operating within the Topos

The monad is used in functional programming for process. The term originates from Leibniz for an elementary 'substance' whose interior cannot be examined. In category theory a monad involves three 'cycles' for GF (GFGFGF) of a free functor F: X → Y providing creativity and an underlying functor G: Y → X enforcing the rules, where F and G are adjoint. GF is an endofunctor as its source and target categories (X) are the same. The process, a snap rather than

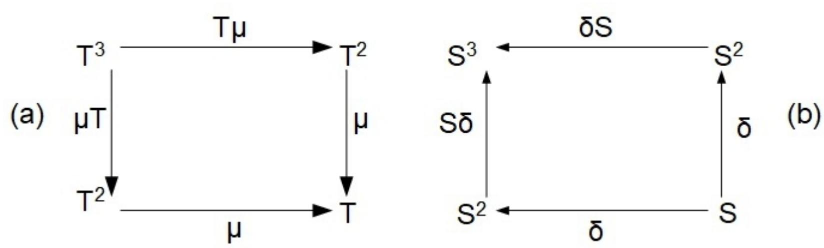

Figure 2. (a) the monad construction $T^3 \rightarrow T^2 \rightarrow T$ where T = GF, μ is multiplication; (b) the comonad construction $S \rightarrow S^2 \rightarrow S^3$ where S = FG, δ is comultiplication

three cycles in turn, emulates a transaction in database systems. Writing GF as T, the monad is shown in Figure 2(a) as $T^3 \rightarrow T^2 \rightarrow T$. There is a dual comonad shown in Figure 2 (b) as $S \rightarrow S^2 \rightarrow S^3$ where S = FG, S^3 is FGFGFG. The monad <T, η, μ> operates within the topos E as T: E → E, and the comonad <S, ε, δ> as S: E → E, where η and ε are the unit and counit of adjunction respectively, and μ and δ are multiplication and comultiplication. A single monad operation takes the score forward just one instance along the time-frame. Cartesian monads are composed naturally to complete a performance, using the Kleisli lift as in implementations in the Haskell functional programming language. Our end-product bears some similarity to the denotators described earlier but uses the topos and the monad to represent process at a conceptual level, more suited for further development of the topic and for discussion with musicians.

6. Anticipation in the Monad/Comonad Structure

There are two distinct viewpoints of anticipation in music: the performer's and the listener's. The player anticipates the sequence of notes to come by extrapolating intuitively from preceding notes while at the same time physically keeping an eye on the subsequent score and the listener builds up a mental image of the music as it evolves. Tension builds for listeners as their anticipation of the performance is realised or denied.

Such experience is captured naturally by the monad/comonad structure with its forward/backward nature and inherent adjointness. Overall the monad looks backwards ($T^3 \rightarrow T^2 \rightarrow T$) and its comonad forwards ($S \rightarrow S^2 \rightarrow S^3$) in their three cycles. However, the situation is more subtle than this: in each cycle the monad looks forwards (F) and then backwards (G) and its comonad looks backwards (G) and then forwards (F). The duality of the monad/comonad represents communication in an orderly manner within initially defined co-limits and adjointness. Values for η (unit) and ε (counit) represent rhetoric and dialectic respectively for the performance, giving a measure of expressiveness and intonation. It is possible that there is a faltering in the communication, resulting in a roll-back with revised adjointness. This remains an area of active study.

Acknowledgements: Members of the Royal Northern Sinfonia,The Sage, Gateshead, UK, especially Alexandra Raikhlina, Sub-Principal 1st violin.

7. References

Klumpenhouwer, Henry, (1991); A Generalized Model of Voice-Leading for Atonal Music, PhD Thesis, Harvard University.

Lewin, David, (2002); Thoughts on Klumpenhouwer Networks and Perle-Lansky Cycles, Music Theory Spectrum **24**(2) 196-230.

Mazzola, Guerino, (2002); The Topos of Music - Geometric Logic of Concepts, Theory, and Performance, Birkhäuser, Basel, 1335pp.

Mazzola, Guerino, & Andreatta, Moreno, (2006); From a Categorical Point of View: K-Nets as Limit Denotators, Perspectives New Music **44**(2) 88-113.

Popoff, Alexandre, Andreatta, Moreno & Ehresmann, Andrée, (2015); A Categorical Generalization of Klumpenhouwer Networks, 5th Int Conf MCM 303-314.

Popoff, Alexandre, Agon, Carlos, Andreatta, Moreno & Ehresmann, Andrée, (2016); From K-Nets to PK-Nets: A Categorical Approach, Perspectives New Music **54**(2) 5-63.

Popoff, Alexandre, Andreatta, Moreno & Ehresmann, Andrée, (2016); Relational PK-Nets for Transformational Music Analysis, arXiv:1611.02249, 19pp, submitted.

Rossiter, Nick, Heather, Michael & Brockway, Michael, (2018); Monadic Design for Universal Systems, ANPA 37-38, Anton L Vrba (ed.), 369-399.