

Anticipation in Communication: The Example of the Cartesian Monad in Music

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Outline of Presentation 1

- Taking on the challenge of a testing application for the Cartesian monad (categorical) approach to universal design
 - Monad = process, operating on a topos
 - Topos = structure, Cartesian (product)
- Look at previous work using category theory with music

Outline of Presentation 2

- The need for natural application
 - Music is a composition of notes, with rules
 - Category theory is a composition of arrows, with rules

Use top levels of category theory

- With closure over three levels
- Maximise expressiveness in data structuring
 - With topos and recursive intension/extension layers
- Capture the process of performing music
 - With monad, operating within the topos

Earlier Work in Music with Category Theory and Nets 1

- Henry Klumpenhouwer/David Lewin (1991-2002)
 - K-nets and L-nets
 - Transformations from one pitch class to another
 - Graphical technique, using isographs
 - Classical harmony based on Z_{12}

Earlier Work in Music with Category Theory and Nets 2

- Guerino Mazzola (2002)
 - The Topos of Music
 - Develops functorial denotators, based on K-nets
 - Digraphs, graphical structures with edges from one node (pitch class) to another
 - Digraphs permit loops
 - Provides detailed exposition of theory of music
 - Useful for underlying musical structures
 - Rather disjoint treatment of title

Earlier Work in Music with Category Theory and Nets 3

- Guerino Mazzola and Moreno Andreatta (2006)
 - Develop denotators idea
 - Again pitch classes initially based on Z_{12}
 - Vertices are pitch classes; edges are transpositions
 - Digraph is constructed
 - Extend pitch classes with a 4-tuple for articulation: $\langle \text{onset}, \text{pitch}, \text{loudness}, \text{duration} \rangle$
 - Use powerobjects to represent chords
 - Paths are maintained through the music
 - Work appears to be based on the Eilenberg-Moore category: the pullback of the category of presheaves on the Kleisli category along the Yoneda embedding (not cited as such)

Earlier Work in Music with Category Theory and Nets 4

- Alexandre Popoff, Carlos Agon, Moreno Andreatta, Andrée Ehresmann (2016a)
 - Developed generalised PK-nets (poly-K)
 - Giving more flexibility in structure of nodes
 - Variable cardinalities
 - Labelling flexible for different genre
 - Defining a natural transformation between functors to achieve the flexibility
- Shortly after (2016b), they introduced the REL category for relationships within nodes, replacing SET

Feelings on Earlier Work

- Sound advance in basic musical structures
- But some of the work appears to be categorification
 - Direct 1:1 translation from set (graph) theory
- And there may be more natural methods for
 - Composition
 - Data structuring with topos
 - Process or communication
- That on denotators in 2006 (Guerino Mazzola and Moreno Andreatta) with apparent use of the Eilenberg-Moore category comes closest to the ideas presented here
- And the PK-nets or denotators could be used as a representation at an underlying level for the score, helping higher-level workers

The Aim is a Topos – Structural Data-type

- Based on Locally Cartesian Closed Category (LCCC) [Descartes]
 - relationships within a product (pullbacks, limits)
 - connectivity (exponentials)
 - internal logic (λ -calculus)
 - identity (from the limit)
 - interchangeability of levels (object to category-object)
 - hyperdoctrine (adjointness between quantifiers and the diagonal)
 - categories held at lower levels are also pullbacks

The Aim is a Topos – Structural Data-type 2

- If we add:
 - definition of relationships within a coproduct (colimits)
 - internal intuitionistic logic (Heyting)
 - subobject classifier (query)
 - reflective subtopos viewpoint (query closure)
- We get a Topos [Aristotle]

A Topos for Music

- Music is viewed as a communication of some manuscript by communicators
- The topos is relatively static (compared to the monad) but being arrow-based can readily handle change.
- Manuscript comprises scores and other intentions of composers and writers
 - Includes musical notation (typeset, handwritten or digital) or more spontaneous formats
- Communicators comprise performers and other aspects of performance
 - Includes an orchestra, group, recording company

Intension/Extension [Aristotle]

- Arguably the most important feature in music
 - Terms come from philosophy
- In mathematics/computing science:
 - The intension is the type, the extension is the collection of instances that satisfy the type
- It's not as simple though as a hierarchy of types
 - There remains a philosophical dimension

Universe of Discourse

- The Universe contains everything
- The Universe of Discourse (UoD) is that section of the Universe of interest to our application
- By the laws of physics we cannot isolate any part of the Universe but we can identify a section for our work
- In this case
 - The intension is the Universe
 - The extension is the world of music (UoD)

A Score is far from fixed

- A musical manuscript has both
 - Intensional properties
 - as a type for how the work is to be performed
 - according to the composer
 - Extensional properties
 - as instances of the manuscript
 - according to variants in
 - Publication (composer initiated, developments after composer's death)
 - Rehearsal (conductor initiated)
 - Performance
 - No two performances are ever the same

A Manuscript is both extensional and intensional

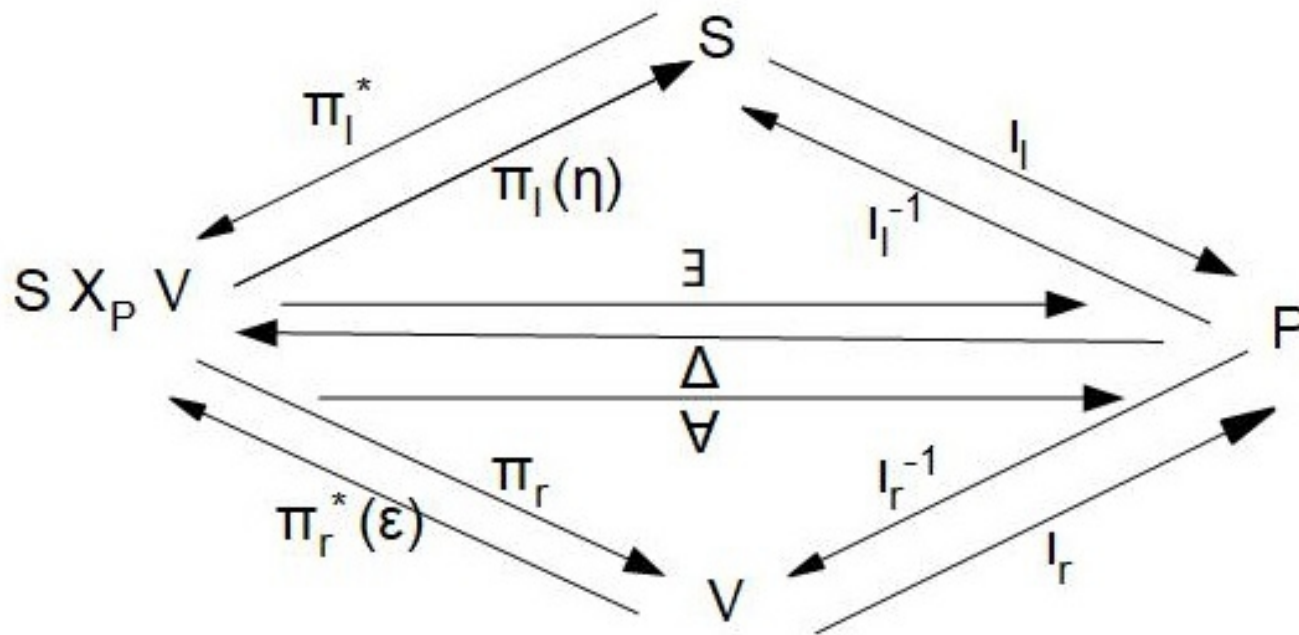
- A musical manuscript is extensional to the Universe of Discourse of Music
 - One of the objects in this universe
- But intensional to the manuscript, its variants and their performances
 - Defining the underlying objects
- So elaborate intension/extension hierarchies can be constructed
 - Where there is a genuine semantic change
 - Category theory suggests four levels are adequate

Pullbacks for Relationships

- In category theory relationships can be represented by:
 - Products (unqualified X)
 - Pullbacks (qualified X)
 - Union (unqualified $+$)
 - Pushouts (qualified $+$)
- The pullback for the Manuscript/ Variant/ Performance relationship follows

Relationship of Score by Variant in Context of Performance ($S \times_P V$)

- Pullback – Locally Cartesian Closed Category



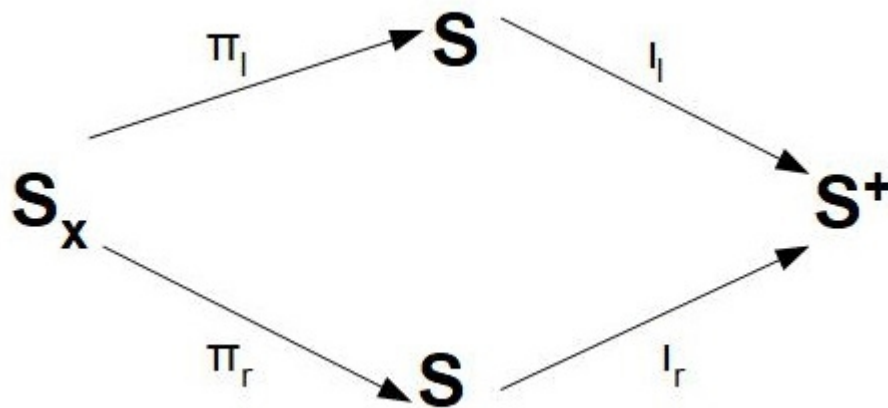
S is category for Score, V for Variant, P for Performance,
 $S \times_P V$ for the relationship

Realising the Extensional Part

- Preceding diagram appears to be the intensional structure
 - The definition (or type structure)
- There is also the extension
 - The instances (conforming to the type structure)
- The diagram actually does include the extension as well, within the category-objects of the pullback $S \times_P V$, S , P , V

Intension/Extension Relationship for Category-object S

- Type/Instance as Dolittle Diagram



As π_l is monic, then so is I_r : diagram is both a pullback and a pushout with limit, product, colimit, coproduct

It's an adhesive category, also known as a pulation square

S is score; top S is intension (type); bottom S is extension (set-values); S_x is limit (type \times value pairs); S^+ is colimit (type + value pairs)

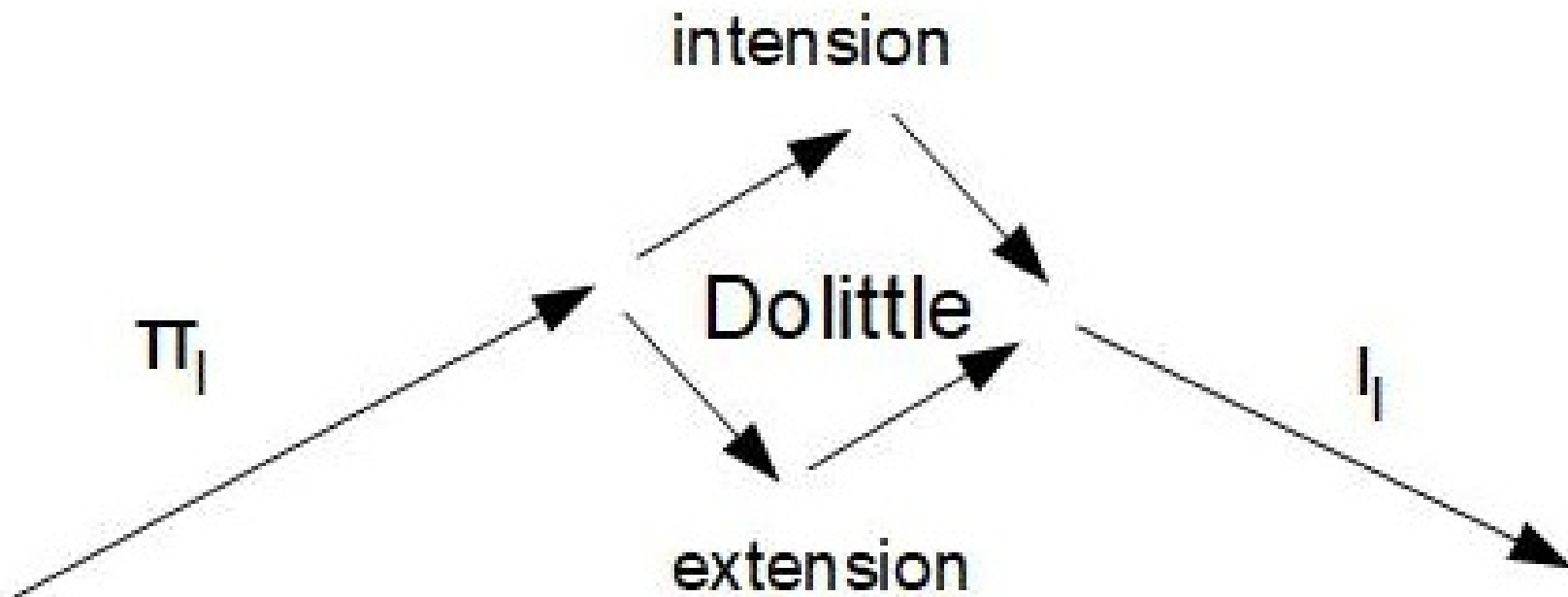
The extension for the score will contain the notes, perhaps as digraphs

Intension/Extension and the Topos

- Every node (category-object) in our pullback relationships will contain such an:
 - Intension part
 - Extension part
- in a Dolittle structure
 - also known as Pulation square, Adhesive category
- Adhesive categories are readily embedded into a topos \mathcal{E}

Category-object Expanded

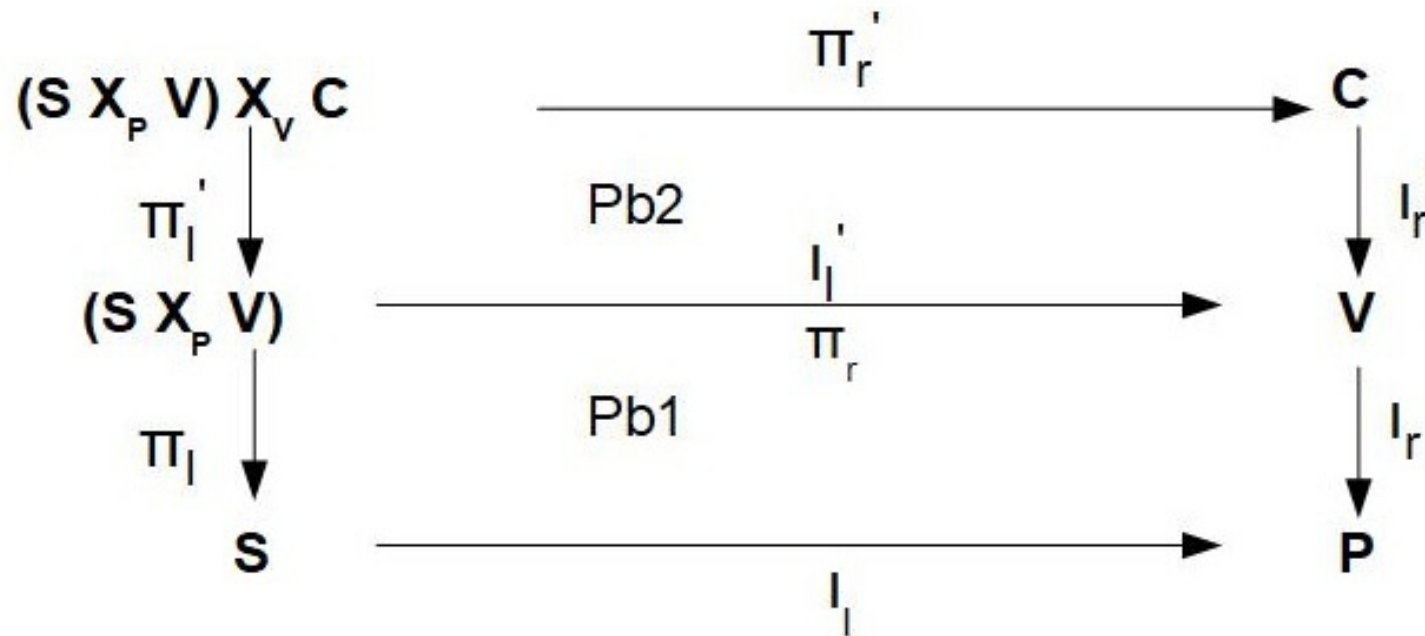
- Category-object S expanded in pullback $S \times_P V$



Data Structuring with Pullbacks

- In real-world, nodes contain more structure than shown
- Also the real-world is more complex than one pullback.
- Need to build more complex structures than that shown. Could:
 - Expand category-objects with further levels
 - Inherent in Locally Cartesian Closed Category approach
 - Paste pullbacks together
 - Pursued in information systems with satisfactory results
- This is still an experimental area

Example of Pasted Pullback



Have 3 pullbacks: $Pb2 \times Pb1$; $Pb2$; $Pb1$

C is category-object for Composer

Overall relationship is of Score with Variant and Composer in context of Performance

Process within the Topos

- Philosophy
 - Metaphysics (Whitehead *Process and Reality* 1929)
 - All is flux [Heraclitus]
- Transaction (Universe or information system)
- Activity
 - Can be very complex but the whole is viewed as atomic – binary outcome – succeed or fail
 - Before and after states must be consistent in terms of rules
 - Intermediate results are not revealed to others
 - Results persist after end

Promising Technique – Monad

- Philosophy of Leibniz
 - Elementary 'substance' whose interior cannot be examined (encapsulation)
- The monad is used in pure mathematics for representing process
 - Has 3 'cycles' of iteration to give consistency

Monad in Functional Programming

- The monad is used to formulate the process in an abstract data-type
- In the Haskell language the monad is a first-class construction
 - Haskell B Curry transformed functions through currying in the λ -calculus
 - The Blockchain transaction system for Bitcoin and more recently other finance houses uses monads via Haskell
 - Reason quoted: it is simple and clean technique
 - Shortage of Haskell programmers has encouraged the use of the Python language

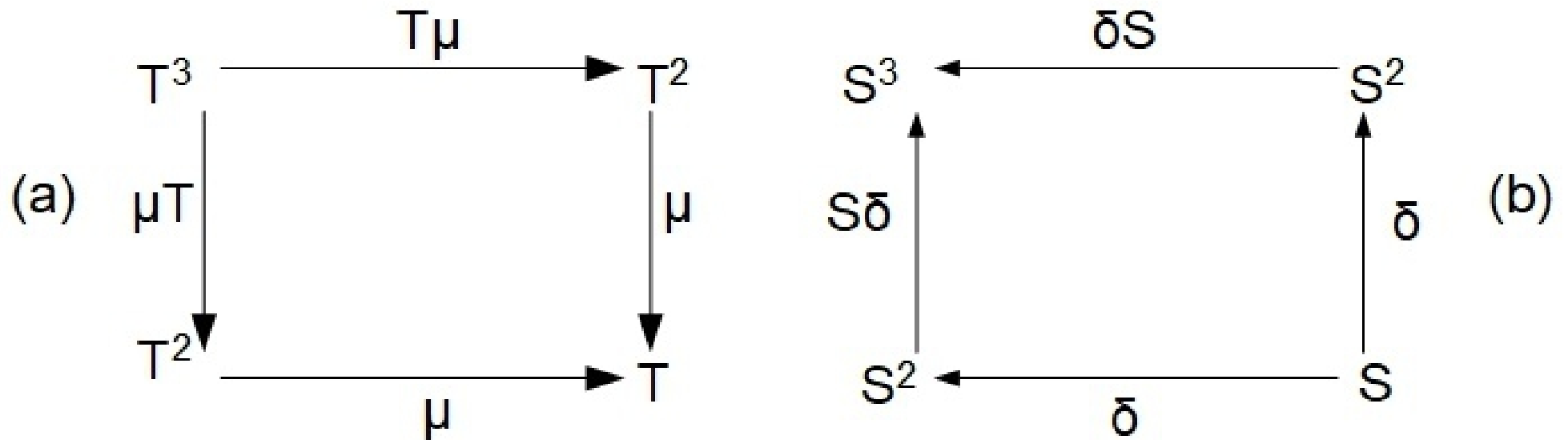
Monad can be based on an adjunction

- The transaction involves GF, a pair of adjoint functors $F \dashv G$
 - $F: X \rightarrow Y$
 - $G: Y \rightarrow X$
- GF is an endofunctor as category X is both source and target
- So T is GF (for monad)
- And S is FG (for comonad)

Monad/Comonad Overview

- Functionality for free functor T , underlying functor S
 - Monad
 - $T^3 \rightarrow T^2 \rightarrow T$ (multiplication)
 - 3 'cycles' of T (GFGFGF)
 - Comonad (dual of monad)
 - $S \rightarrow S^2 \rightarrow S^3$ (comultiplication)
 - 3 'cycles' of S (FGFGFG)
- Objects
 - An endofunctor on category \mathcal{C} (the topos)
- Note this multiple performance matches our transaction approach, outlined earlier, with GF performed 3 times

Monad/Comonad Diagrams

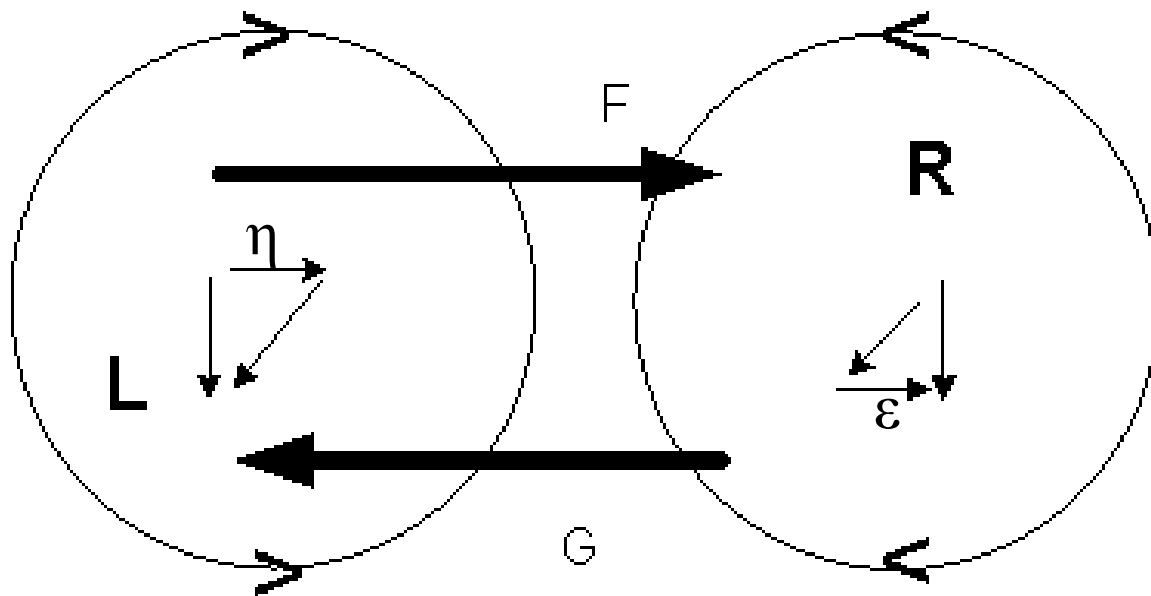


(a) the monad construction $T^3 \rightarrow T^2 \rightarrow T$ where $T = GF$,
multiplication = μ

(b) the comonad construction $S \rightarrow S^2 \rightarrow S^3$ where $S = FG$,
comultiplication = δ

A 'Cycle' representing adjointness

- One 'cycle' for monad T (GF)
 - Assessing unit η in L and counit ε in R to ensure overall consistency
 - One or more 'Cycles' is performed simultaneously (a snap, not each cycle in turn).



$F \dashv G$

$$\eta: 1_L \rightarrow GF(L)$$

$$\varepsilon: FG(R) \rightarrow 1_R$$

Failure in Expected Adjointness

- Means transaction has failed
- Communication is suspended
- Restart is necessary at some convenient point (Rollback)
- In music need to distinguish between a wrong note and differences in expression:
 - Intonation is the rules, on violin with left hand (left exact)
 - Articulation is the expression, on violin with right hand (right exact)
- Failure leads to revised adjointness

Operating within a Topos

- The monad operation is simple:
 - $T: \mathcal{E} \rightarrow \mathcal{E}$
 - where T is the monad $\langle GF, \eta, \mu \rangle$ in \mathcal{E} , the topos, with input and output types the same
- The extension (data values) will vary but the intension (definition of type) remains the same
- Closure is achieved as the type is preserved

Process in Musical Performance

- The topos \mathcal{E} created earlier contains
 - The intension/extension in the categories S (for Score, with musical notation), V (for Variant), $S \times_P V$ (for their qualified product), P (for performance for the actual musical event)
- A single monad/comonad action (of 3 cycles T^3) will take the music forward one unit of performance (phrase or bar), say one step

Process in Musical Performance 2

- Moving from one barline to another is determined uniquely by the adjunction $F \dashv G$
 - F is the free functor (looking forward, creative/expressive)
 - G is the underlying functor (looking back, enforcing the rules, qualia)

Process in Musical Performance 3

- If adjointness holds over the 3 cycles
 - Then η the unit of adjunction measures the creativity of the step going forward (rhetoric)
 - And ε the counit of adjunction measures the qualia of the step looking back (dialectic)
- If expected adjointness does not hold over the 3 cycles
 - Then integrity has been lost and resynchronization is necessary with revised adjointness

Comparison with Earlier work

- Our end-product bears some similarity to the denotators described earlier but
 - uses the topos and the monad to represent process at a conceptual level
 - is more suited for further development of the topic
 - encouraging discussion with musicians.

Experience

- Performers do comment that playing is an intensive experience:
 - at the same time both looking back as to what you have played and anticipating what is to come.
- Such experience is captured by the monad/comonad structure with its forward/backward nature and inherent adjointness

Monad/Comonad Direction

- Overall the monad looks backwards
 - $T^3 \rightarrow T^2 \rightarrow T$
- and its comonad forwards
 - $S \rightarrow S^2 \rightarrow S^3$
- in their three cycles.
- However, the situation is more subtle than this:
 - in each cycle the monad looks forwards (F) and then backwards (G)
 - its comonad looks backwards (G) and then forwards (F)

Orderly Communication

- The duality of the monad/comonad represents communication in an orderly manner within initially defined colimits and adjointness.

Composition

- A musical work is referred to as a composition.
- It is indeed a composition of steps
 - With the output from one step becoming the input to the next step
- The order is fixed in advance
- Composition is an inherent feature of category theory
- With one monad execution as a single step, it is necessary to compose monads to perform a full work

Therefore composability is the Key

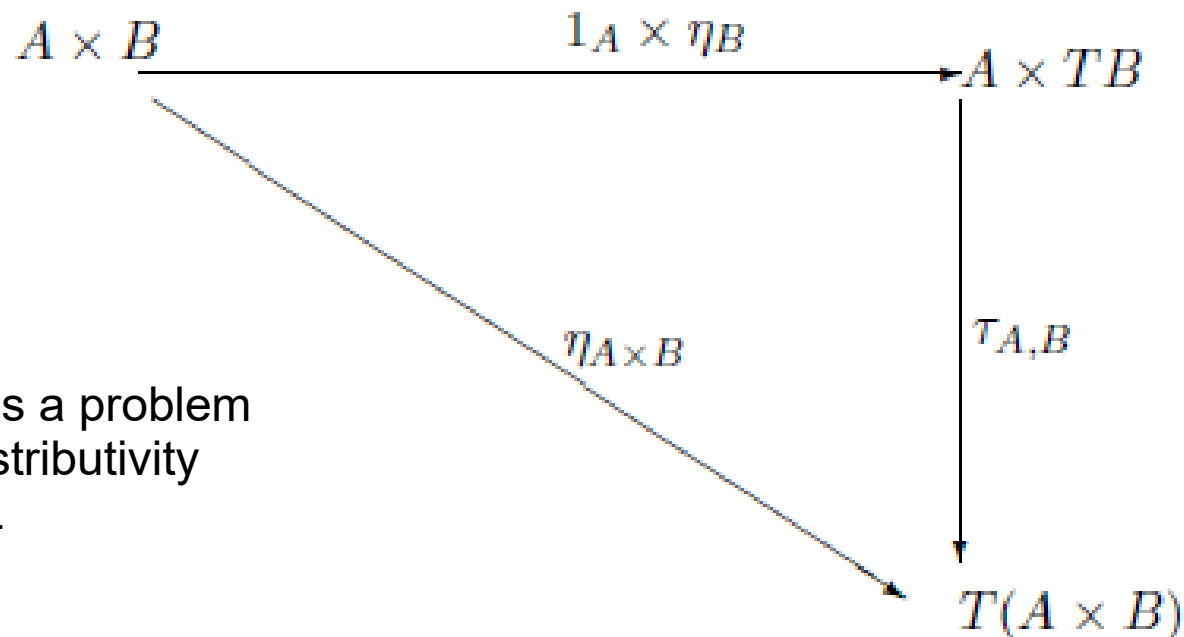
- Compose many monads together to give the power of adjointness over a whole wide-ranging application
- In banking (Bitcoin) the reliability obtained from composing processes over a wide-range of machines (distributed data recovery) justifies the move to Category Theory
- There is a problem though in EML (Eilenberg/Mac Lane) Category Theory:
 - Monads do not compose naturally

Haskell and Monads

- Kleisli Category of a Monad
 - Transforms a monad into a monadic form more suitable for implementation in a functional language
 - Used in Haskell rather than the pure mathematics form of Mac Lane
- Strengthens the monad for composability
 - As in the Cartesian Monad, with products
- A practical application of the pure maths has exposed problems in the maths
- Solution has come from another pure mathematician Kleisli

Kleisli Lift

- Define a natural transformation:
 - $\tau_{A,B} : A \times TB \rightarrow T(A \times B)$ where A, B are objects in X and T is the monad such that the following diagram commutes



There is a problem
with distributivity
In EML

Cartesian Monads in Music

- Take each barline, or some other time signature, as a unit of process
 - Such a barline will be Cartesian, representing the potentially complex physics of the music
 - Combinations of notes, including chords
 - Or powerobjects as in the denotators approach
- Therefore Cartesian Monads as strengthened by the Kleisli Lift are essential for composition purposes

Anticipation in the Monad/Comonad Structure

- There are two distinct viewpoints of anticipation in music: the performer's and the listener's.
- The player anticipates the sequence of notes to come by extrapolating intuitively from preceding notes while at the same time physically keeping an eye on the subsequent score and the listener builds up a mental image of the music as it evolves.
- Tension builds for listeners as their anticipation of the performance is realised or denied.

Anticipation is Natural with the monad

- Such experience is captured naturally by the monad/comonad structure with its forward/backward nature and inherent adjointness.
- Overall the monad looks backwards ($T^3 \rightarrow T^2 \rightarrow T$) and its comonad forwards ($S \rightarrow S^2 \rightarrow S^3$) in their three cycles.
- However, the situation is more subtle than this: in each cycle the monad looks forwards (F) and then backwards (G) and its comonad looks backwards (G) and then forwards (F).

Orderly Communication

The duality of the monad/comonad represents communication in an orderly manner within initially defined co-limits and adjointness.

Values for η (unit) and ε (counit) represent rhetoric and dialectic respectively for the performance, giving a measure of expressiveness and intonation.

It is possible that there is a faltering in the communication, resulting in a roll-back with revised adjointness. This remains an area of active study.

Preliminary View from the Mind

- With example of violin
 - Left-hand performs pitch control (intonation)
 - Right-hand performs bowing (articulation)
- Hemisphere of brain
 - Left-hand maps into rhs (intonation)
 - Right-hand maps into lhs (articulation)
- Adjointness between rhs and lhs
- Monad controls adjointness as move through score
- Musicians who learnt early in life have
 - increased plasticity of the corpus callosum during a sensitive period of time in development
 - to accommodate the monad

Summary of Progress/Look forward

- Topos has been established as data-type of choice
- Monad shows potential for processing the topos
- There is no assumption of any particular musical genre.
- Such a categorial framework could be implemented in the functional programming language Haskell
 - Basic physical music structures have been implemented in Haskell (Paul Hudak)

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