The Universe as Intension: an Expressive Dynamic Information and Communication System as Extension

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Further Work since last August

- Publication ANPA of information systems paper
- Literature review, music, nets and categories
- Fieldwork with orchestra
- Further presentations
  - UNILOG 2018, Vichy, Workshop: Logic and Music
  - Anticipative Systems, Baden-Baden
- Technical issues
  - Data structures section revised, intension/extension
  - Process section revised, brain activity
  - Anticipation introduced

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  - Alexandra Raikhlina, Sub-principal 1st violin
- For discussions on pure category theory
  - Michael Brockway

Outline of Presentation 1

- Taking on the challenge of a testing application for the Cartesian monad (categorical) approach to universal design
  - Monad = process, operating in a topos
  - Topos = structure, Cartesian (product)
- Look at previous work using category theory with music

Outline of Presentation 2

- The need for natural application
  - Music is a composition of notes, with rules
  - Category theory is a composition of arrows, with rules

  Use top levels of category theory
    - With closure over three levels
  - Maximise expressiveness in data structuring
    - With topos and recursive intension/extension layers
  - Capture the process of performing music
    - With monad, operating within the topos

Earlier Work in Music with Category Theory and Nets 1

- Henry Klumpenhouwer/David Lewin (1991-2002)
  - K-nets and L-nets
  - Transformations from one pitch class to another
  - Graphical technique, using isographs
  - Classical harmony based on Z_{12}

Earlier Work in Music with Category Theory and Nets 2

- Guerino Mazzola (2002)
  - The Topos of Music (1338pp)
  - Develops functorial denotators, based on K-nets
    - Digraphs, graphical structures with edges from one node (pitch class) to another
    - Digraphs permit loops
  - Provides detailed exposition of theory of music
    - Useful for underlying musical structures
  - Rather disjoint treatment of title

Earlier Work in Music with Category Theory and Nets 3

- Guerino Mazzola and Moreno Andreatta (2006)
  - Develop denotators idea
    - Again pitch classes initially based on Z_{12}
    - Vertices are pitch classes; edges are transpositions
    - Digraph is constructed
    - Extend pitch classes with a 4-tuple for articulation: <onset, pitch, loudness, duration>
    - Use powerobjects to represent chords
    - Paths are maintained through the music
  - Work appears to be based on the Eilenberg-Moore category: the pullback of the category of presheaves on the Kleisli category along the Yoneda embedding (not cited as such)

Earlier Work in Music with Category Theory and Nets 4

- Alexandre Popoff, Carlos Agon, Moreno Andreatta, Andrée Ehresmann (2016a)
  - Developed generalised PK-nets (poly-K)
    - Giving more flexibility in structure of nodes
      - Variable cardinalities
      - Labelling flexible for different genre
    - Defining a natural transformation between functors to achieve the flexibility
- Shortly after, (2016b), they introduced the REL category for relationships within nodes, replacing SET

Feelings on Earlier Work

- Sound advance in basic musical structures
- But some of the work appears to be categorification
  - Direct 1:1 translation from set (graph) theory
- And there may be more natural methods for
  - Composition
  - Data structuring with topos
  - Process or communication
- That on denotators in 2006 (Guerino Mazzola and Moreno Andreatta) with apparent use of the Eilenberg-Moore category comes closest to the ideas presented here
- And the PK-nets or denotators could be used as a representation at an underlying level for the score, helping higher-level workers

The Aim is a Topos – Structural Data-type

- Based on Locally Cartesian Closed Category (LCCC) [Descartes]
  - relationships within a product (pullbacks, limits)
  - connectivity (exponentials)
  - internal logic (λ-calculus)
  - identity (from the limit)
  - interchangeability of levels (object to category-object)
  - hyperdoctrine (adjointness between quantifiers and the diagonal)
  - categories held at lower levels are also pullbacks

The Aim is a Topos – Structural Data-type 2

- If we add:
  - definition of relationships within a coproduct (colimits)
  - internal intuitionistic logic (Heyting)
  - subobject classifier (query)
  - reflective subtopos viewpoint (query closure)
- We get a Topos [Aristotle]

A Topos for Music

- Music is viewed as a communication of some manuscript by communicators
- The topos is relatively static (compared to the monad) but being arrow-based can readily handle change.
- Manuscript comprises scores and other intentions of composers and writers
    - Includes musical notation (typeset, handwritten or digital) or more spontaneous formats
- Communicators comprise performers and other aspects of performance
    - Includes an orchestra, group, audience, recording company

Intension/Extension [Aristotle]

- Arguably the most important feature in music
  - Terms come from philosophy
- In mathematics/computing science:
  - The intension is the type, the extension is the collection of instances that satisfy the type
- It's not as simple though as a hierarchy of types
  - There remains a philosophical dimension

Universe of Discourse

- The Universe contains everything
- The Universe of Discourse (UoD) is that section of the Universe of interest to our application
- By the laws of physics we cannot isolate any part of the Universe but we can identify a section for our work
- In this case
  - The intension is the Universe
  - The extension is the world of music (UoD)

A Score is far from fixed

- A musical manuscript has both
  - Intensional properties
    - as a type for how the work is to be performed
    - according to the composer
  - Extensional properties
    - as instances of the manuscript
    - according to variants in
      - Publication (composer initiated, developments after composer's death)
      - Rehearsal (conductor initiated)
      - Performance
        - No two performances are ever the same

A Manuscript is both extensional and intensional

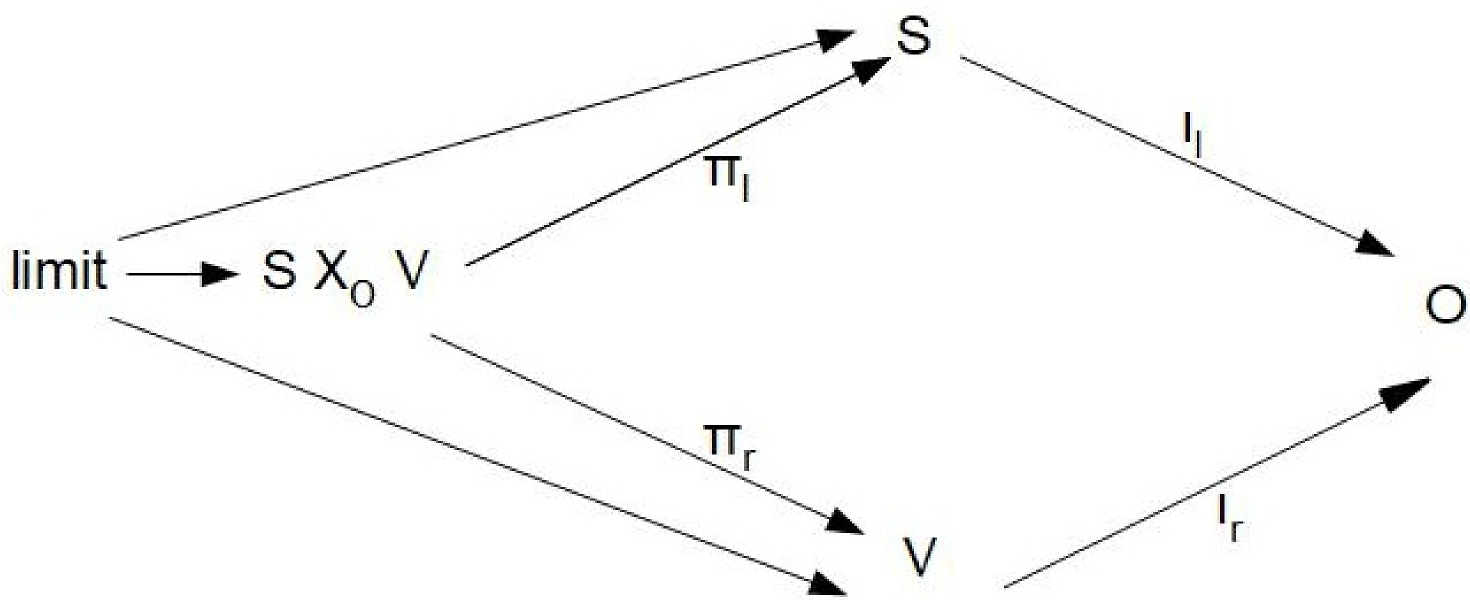
- A musical manuscript is extensional to the Universe of Discourse of Music
  - One of the objects in this universe
- But intensional to the manuscript's variants and performances
  - Defining the underlying objects
- So elaborate intension/extension hierarchies can be constructed
  - Where there is a genuine semantic change
  - Category theory suggests four levels are adequate

Pullbacks for Relationships

- In category theory relationships can be represented by:
  - Products (unqualified X, all possible pairs)
  - Pullbacks (qualified X)
  - Union (unqualified +, disjoint, duplicates not permitted)
  - Pushouts (qualified +, direct sum)
- The simple pullback for the Manuscript/ Variant/ Performance relationship follows

Relationship of Score by Variant in Context of Occasion ($S\ X_O\ V$)

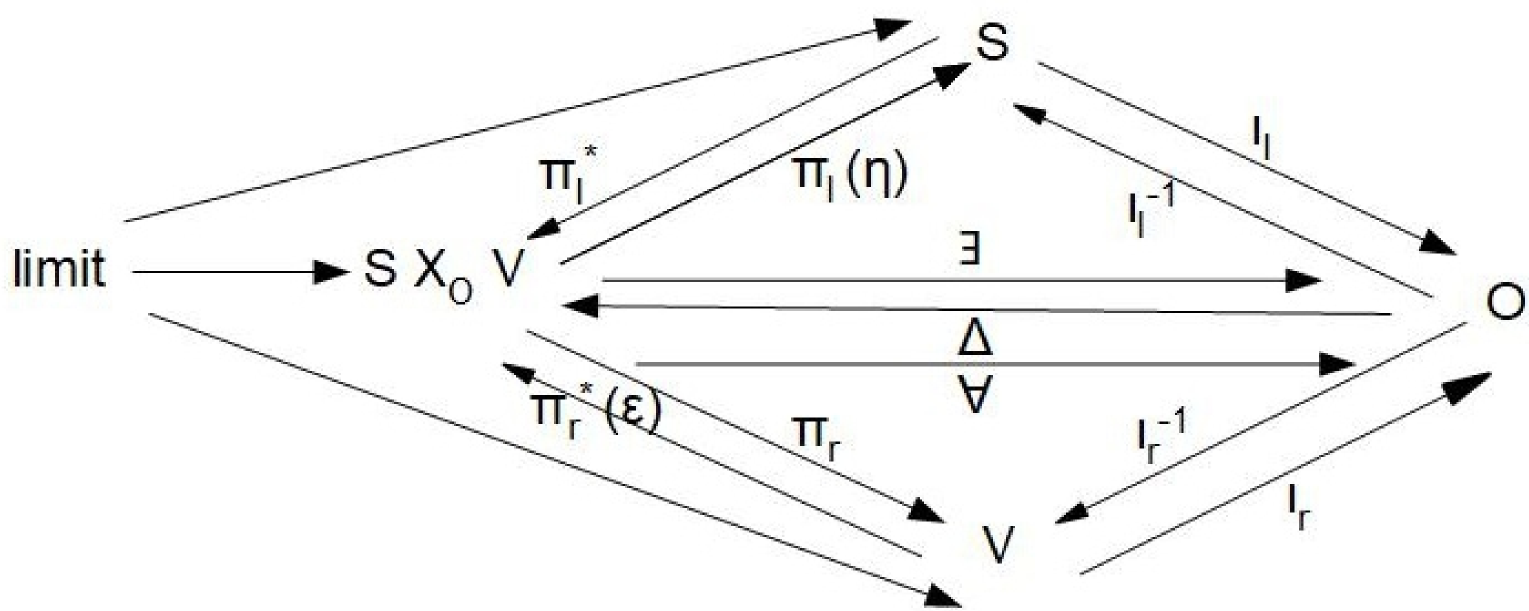
- Simple Pullback – Cartesian Category



S is category for Score, V for Variant, O for Occasion,
$S\ X_O\ V$ for the relationship

Relationship of Score by Variant in Context of Occasion (S X_O V)

- Pullback – Locally Cartesian Closed Category



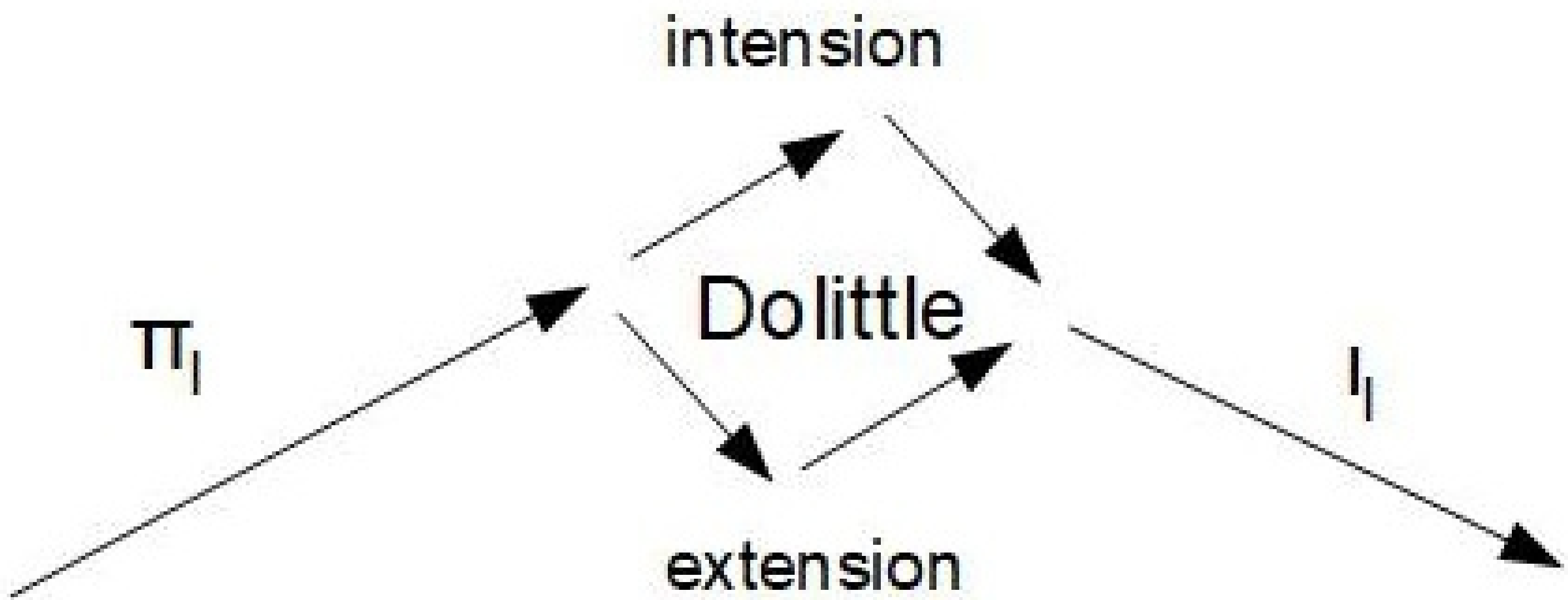
S is category for Score, V for Variant, O for Occasion,
S X_O V for the relationship

Realising the Extensional Part

- Preceding diagram appears to be the intensional structure
  - The definition (or type structure)
- There is also the extension
  - The instances (conforming to the type structure)
- The diagram actually does include the extension as well, within the category-objects of the pullback $S \times_O V$, S, V, O

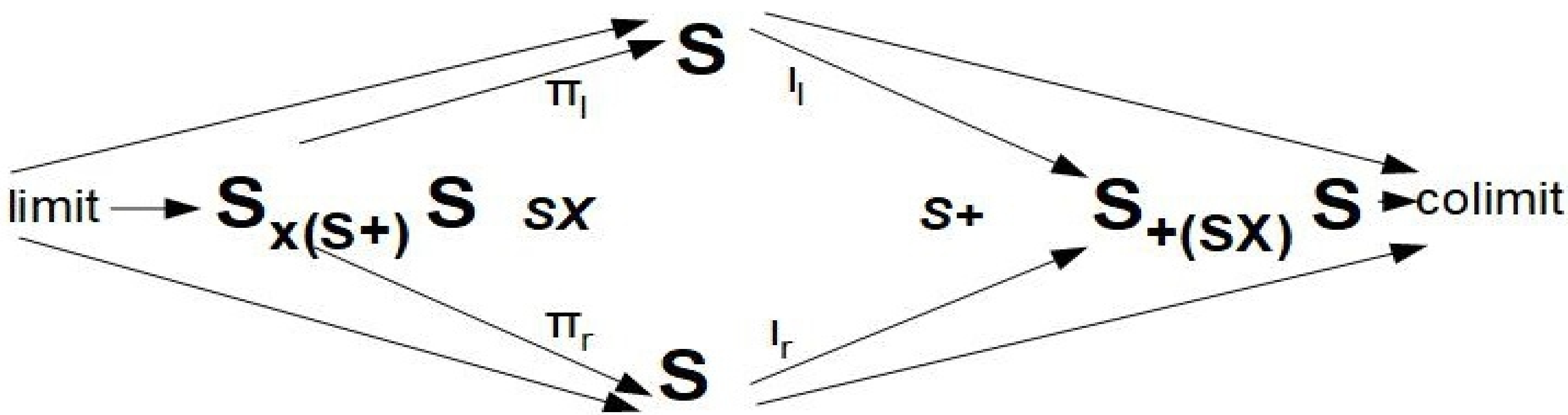
Category-object Expanded

- Category-object S expanded in pullback S X_O V



Intension/Extension Relationship for Category-object S

- Type/Instance as Dolittle Diagram



It is an adhesive category, also known as a pulation square

As π_l is monic, then so is I_r : diagram is both a pullback and a pushout with limit, product, colimit, coproduct

S is score; top S is intension (*type*); bottom S is extension (*set-values*); S_X is limit (*type* X *value* pairs); S^+ is colimit (*type* + *value* pairs)

The extension for the score will contain the notes, perhaps as digraphs

Intension/Extension and the Topos

- Every node (category-object) in our pullback relationships will contain such an:
  - Intension part
  - Extension part
- in a Dolittle structure
  - also known as Pulation square, Adhesive category
- Adhesive categories are readily embedded into a topos Є

Data Structuring with Pullbacks

- In real-world, nodes contain more structure than shown
- Also the real-world is more complex than one pullback.
- Need to build more complex structures than that shown. Could:
  - Expand category-objects with further levels
    - Inherent in Locally Cartesian Closed Category approach
  - Paste pullbacks together
    - Pursued in information systems with satisfactory results
- This is still an experimental area

Example of Pasted Pullback

$$
\begin{tikzcd}
(S\ X_o\ V)\ X_v\ C \arrow[r, "\pi_r'"] \arrow[d, "\pi_l'"'] \arrow[dr, phantom, "\mathrm{Pb2}"] & C \arrow[d, "I_r'"] \\
(S\ X_o\ V) \arrow[r, "I_l'", "\pi_r"'] \arrow[d, "\pi_l"'] \arrow[dr, phantom, "\mathrm{Pb1}"] & V \arrow[d, "I_r"] \\
S \arrow[r, "I_l"'] & O
\end{tikzcd}
$$

Have 3 pullbacks: Pb2 X Pb1; Pb2; Pb1

C is category-object for Composer

Overall relationship is of Score with Variant and Composer in context of Occasion

The Data Structure as a Whole

- Will be the topos PERF, including
  - The Locally Cartesian Closed Category described earlier $S \; X_O \; V$
  - Plus
    - Any decompositions of category-objects (vertical)
    - Any pastings onto the LCCC (horizontal)
- This topos will be a placeholder for the adhesive categories defined above
  -

Process within the Topos

- Philosophy
  - Metaphysics (Whitehead *Process and Reality* 1929)
  - All is flux [Heraclitus]
- Transaction (Universe or information system)
  - Activity, requires 3 cycles:
    - Can be very complex but the whole is viewed as atomic – binary outcome – succeed or fail
    - Before and after states must be consistent in terms of rules
    - Intermediate results are not revealed to others
    - Results persist after end

Promising Technique – Monad

- Philosophy of Leibniz
  - Elementary 'substance' whose interior cannot be examined (encapsulation)
- The monad is used in pure mathematics for representing process
  - Has 3 'cycles' of iteration to give consistency
- Musical meaning as single note gives scope for confusion

Monad in Functional Programming

- The monad is used to formulate the process in an abstract data-type
- In the Haskell language the monad is a first-class construction
  - Haskell B Curry transformed functions through currying in the λ-calculus
  - The Blockchain transaction system for Bitcoin and more recently other finance houses uses monads via Haskell
  - Reason quoted: it is simple and clean technique
  - Shortage of Haskell programmers has encouraged the use of the Python language

Processes in the Brain

- Use as an indication of
  - Structures required
  - Processes occurring
  - Control of processes

Preliminary View from the Mind

- With example of violin
  - Left-hand performs pitch control (intonation)
  - Right-hand performs bowing (articulation)
- Hemisphere of brain
  - Left-hand maps into rhs (intonation)
  - Right-hand maps into lhs (articulation)
- Coordination between lhs and rhs ('rhythm')
- Control required as move through score
- Musicians who learnt early in life have
  - enhanced corpus callosum, to accommodate the control required of the two hemispheres in the brain

Co-ordination of the Brain in Category Theory

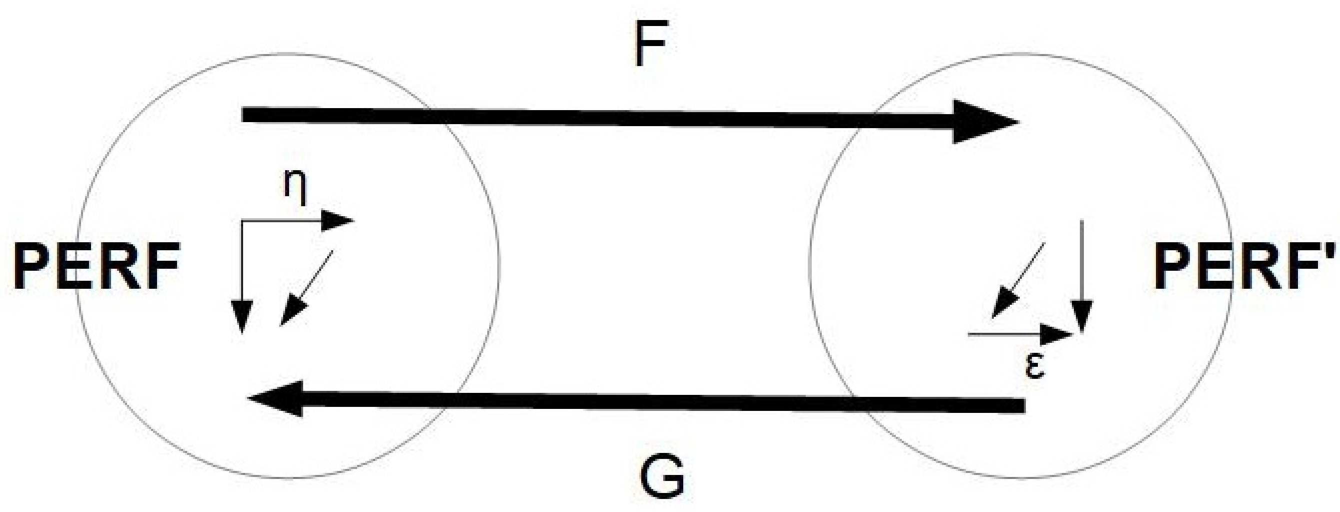
- Need close (natural) relationship (adjointness) between the lhs and rhs
  - The lhs will be an activity, a functor F, representing articulation
    - $F: PERF \rightarrow PERF'$
  - The rhs will be an activity, a functor G, representing intonation
    - $G: PERF' \rightarrow PERF$

Intensive Mental Effort

- F, G will operate on part of the Universe of Discourse of Music
  - identified as a topos
  - for the score and players in question
  - under the control of a monad
- The monad performs the co-ordination through the corpus callosum
- Brain activity pattern is consistent with our monad approach
  - for the violin
  - not necessarily for other instruments

One 'Cycle' for adjointness F -| G

- One ‘cycle’ for GF
  - Assessing unit η in PERF and counit ε in PERF' to ensure overall consistency



η: $1_{PERF} \rightarrow GF(PERF)$ ε: $FG(PERF') \rightarrow 1_{PERF''}$

Monad can be based on an adjunction

- A monad based on an endofunctor
  - Is a functor with the same source and target
- An adjunction F -| G is an endofunctor
  - F: X → Y
  - G: Y → X
- GF is an endofunctor as category X is both source and target
- So a monad T is GF
- And a comonad S, dual of T, is FG
- Our monads are Cartesian (involve products)

Monad/Comonad Overview

- Functionality for free functor T and underlying functor S
  - Monad
    - $T^3 \rightarrow T^2 \rightarrow T$ (multiplication, 3 'cycles' of T (GFGFGF))
  - Comonad (dual of monad)
    - $S \rightarrow S^2 \rightarrow S^3$ (comultiplication, 3 'cycles' of S (FGFGFG))
  - 'cycles' as process is a snap, not a series of cycles
- Objects
  - An endofunctor on category Є (the topos)
- This multiple performance matches our transaction approach, outlined earlier, with GF performed 3 times

Monad/Comonad Diagrams

$$\text{(a)}\quad \begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \qquad\qquad \begin{tikzcd} S^3 & S^2 \arrow[l, "\delta S"'] \\ S^2 \arrow[u, "S\delta"] & S \arrow[l, "\delta"] \arrow[u, "\delta"'] \end{tikzcd}\quad \text{(b)}$$

(a) the monad construction $T^3 \rightarrow T^2 \rightarrow T$ where $T = GF$, multiplication = μ

(b) the comonad construction $S \rightarrow S^2 \rightarrow S^3$ where $S = FG$, comultiplication = δ

Process in Musical Performance

- The topos PERF created earlier contains
  - The illustrated pullback S X_O V, plus further decompositions and pastings described earlier
- A single monad/comonad action (of 3 cycles T^3) will take the music forward one unit of performance (phrase or bar), say one step:
  - T: PERF $\rightarrow$ PERF'
- The extension (data values) will vary but the intension (definition of type) remains the same
- Closure is achieved as the type is preserved

Process in Musical Performance 2

- Moving from one barline to another is determined uniquely by the adjunction F -| G
  - F is the free functor (looking forward, creative/expressive)
  - G is the underlying functor (looking back, enforcing the rules, qualia)

Process in Musical Performance 3

- If adjointness holds over the 3 cycles
  - Then η the unit of adjunction measures the creativity of the step going forward (rhetoric)
  - And ε the counit of adjunction measures the qualia of the step looking back (dialectic)
- If expected adjointness does not hold over the 3 cycles
  - Then integrity has been lost and resynchronization is necessary with revised adjointness

Process in Musical Performance 4

- Tolerance has to be given on the acceptable limits and colimits of a performance
- In music need to distinguish between a wrong note and differences in expression
- Failure leads to revised adjointness and possible resynchronization after a rollback to a safe position for a restart

Comparison with Earlier work

- Our end-product bears some similarity to the denotators described earlier but
  - uses the topos and the monad to represent data and process respectively at a conceptual level
  - is more suited for further development of the topic
  - encourages discussion with musicians
  - has a pathway to an implementation

Experience

- Performers do comment that playing is an intensive experience:
  - at the same time both looking back as to what you have played and anticipating what is to come.
- Such experience is captured by the monad/comonad structure with its forward/backward nature and inherent adjointness

Composition

- A musical work is referred to as a composition.
- It is indeed a composition of steps
  - With the output from one step becoming the input to the next step
- The order is fixed in advance
- Composition is an inherent feature of category theory
- With one monad execution as a single step, it is necessary to compose monads to perform a full work

Therefore composability is the Key

- Compose many monads together to give the power of adjointness over a whole wide-ranging application
- In banking (Bitcoin) the reliability obtained from composing processes over a wide-range of machines (distributed data recovery) justifies the move to Category Theory
- There is a problem though in EML (Eilenberg/ Mac Lane) Category Theory:
  - Cartesian monads do not compose naturally

Haskell and Monads

- Kleisli Category of a Monad
  - Transforms a monad into a monadic form more suitable for implementation in a functional language
    - Used in Haskell rather than the pure mathematics form of Mac Lane
- Strengthens the monad for composability
  - As required for the Cartesian monad
- A practical application of the pure maths has exposed problems in the maths
- Solution has come from another pure mathematician Kleisli

Kleisli Lift

- Define a natural transformation:
  - $\tau_{A,B}$: A X TB → T (A X B) where A,B are objects in X and T is the monad such that the following diagram commutes

$$
\begin{tikzcd}
A \times B \arrow[r, "1_A \times \eta_B"] \arrow[dr, "\eta_{A \times B}"] & A \times TB \arrow[d, "\tau_{A,B}"] \\
& T(A \times B)
\end{tikzcd}
$$

There is a problem
with distributivity
In EML

Cartesian Monads in Music

- Take each barline, or some other time signature, as a unit of process
  - Such a barline will be Cartesian, representing the potentially complex physics of the music
    - Combinations of notes, including chords
    - Or powerobjects as in the denotators approach
- Therefore Cartesian Monads as strengthened by the Kleisli Lift are essential for composition purposes

Anticipation in the Monad/Comonad Structure

- There are two distinct viewpoints of anticipation in music: the performer's and the listener's.
- The player anticipates the sequence of notes to come by extrapolating intuitively from preceding notes while at the same time physically keeping an eye on the subsequent score and the listener builds up a mental image of the music as it evolves.
- Tension builds for listeners as their anticipation of the performance is realised or denied.

Anticipation is Natural with the Monad

- Such experience is captured naturally by the monad/comonad structure with its forward/backward nature and inherent adjointness.
- Overall the monad looks backwards ($T^3 \rightarrow T^2 \rightarrow T$) and its comonad forwards ($S \rightarrow S^2 \rightarrow S^3$) in their three cycles.
- However, the situation is more subtle than this: in each cycle the monad looks forwards (F) and then backwards (G) and its comonad looks backwards (G) and then forwards (F).

Anticipation in Communication

- The duality of the monad/comonad represents communication in an orderly manner within initially defined co-limits and adjointness.
- Values for η (unit) and ε (counit) represent rhetoric and dialectic respectively for the performance, giving a measure of expressiveness and intonation.
- It is possible that there is a faltering in the communication, resulting in a roll-back with revised adjointness. This remains an area of active study.

Summary of Progress/Look forward

- Topos has been established as data-type of choice
- Monad shows potential for processing the topos
- There is no assumption of any particular musical genre.
- Such a categorial framework could be implemented in the functional programming language Haskell
  - Basic physical music structures have been implemented in Haskell (Paul Hudak)

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