Monadic Design for Universal Systems

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Outline of Presentation

- Basic categorical facilities identified for the Universe
  - The Topos (structural data-type)
  - The Monad (process)
- Applying the monad to a topos
  - Alternative Techniques
  - Application
- Discussion

The Topos – Structural Data-type

- Based on Cartesian Closed Category (CCC)
  - Products; Closure at top; Connectivity (exponentials); Internal Logic; Identity; Interchangeability of levels
- If we add:
  - Subobject classifier
  - Internal logic of Heyting (intuitionistic)
  - Reflective subtopos (query closure)
- We get a Topos

Topos: further work identified 2-years ago and now

- Data Process
  - Queries within a topos – use of subobject classifier, particularly with power objects
  - Examples of Heyting intuitionistic logic
  - Applying process to a topos
- Database design
  - Cocartesian approach
  - Pasting of pullbacks
  - Recursive pullbacks
  - Allegories

Progress

Data Process	
Queries within a topos	ANPA 36: Subobject classifier extended to power-objects for generality
Heyting examples, internal logic	Stalled for while but now restarted
External process on a topos	Today's topic, monads
Database Design	
Cocartesian	For normalisation, not now a priority as thought to be categorification below 5NF
Pasting of pullbacks	ANPA 36: Expressed in complex, more realistic design, satisfies 5NF
Recursive pullbacks	ANPA 35: Represent different levels of detail
Allegories	ANPA 36: Explored, not useful in natural IS but significant for interoperability

Normalization

- Match in data design between logical and physical world.
- Many stages of normal forms in set-theoretic relational model:1NF, 2NF, 3NF, BCNF, 4NF, 5NF.
- Only last need concern us here.
- 5NF is also known as Project-Join Normal Form

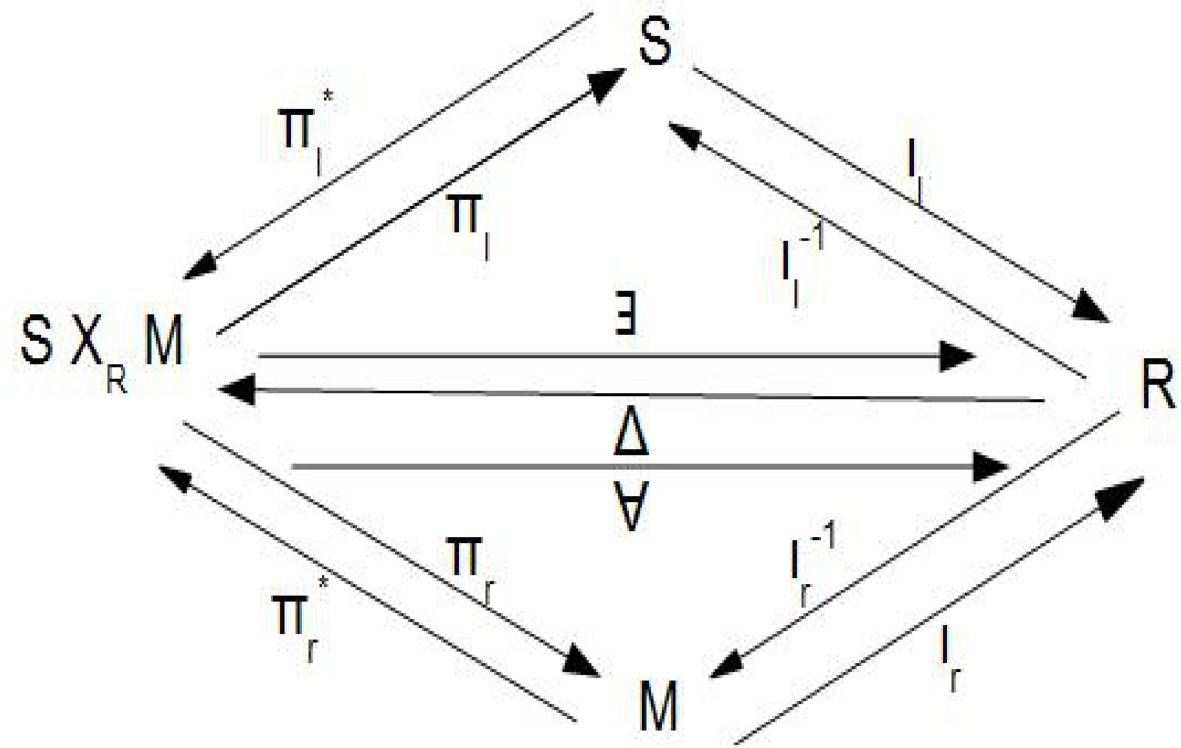
5NF (PJNF) is a Pullback Diagram

- Structure is in 5NF if its projections can be joined to return the original structure.
- Projection is decomposition of a product
- Join is Δ, diagonal from coproduct to product
- Mark Levene & Millist W Vincent evidently consider pullbacks with normalisation
- Simple pullback (two projections, one join) is trivial
- More complex (realistic) examples are handled by pasted pullbacks

Examples

- Student Marks
  - Simple PJ/NF (single pullback)
- Bank Transactions
  - Simple PJ/NF (single pullback)
  - Simple pasted PJ/NF (2 pasted squares, 3 pullbacks)
  - Complex PJ/NF (5 pasted squares, 10 pullbacks)
  - Complex structure (5 pasted squares, not valid pullback)

Pullback - Single Relationship
Student Marks



Pullback - Single Relationship Constraints

- SX_R M (Student X_{Result} Mark)
- Logic of adjointness: $\exists \dashv \Delta \dashv \forall$
  - Δ selects pairs of S and M in a relationship in context of R
  - Such that $\exists \dashv \Delta$ and $\Delta \dashv \forall$
  - Termed by Lawvere as a hyperdoctrine
- Projections π are from product, left and right (dual π^*)
- Inclusions ι are into sum S+M+R, left and right (dual ι^{-1})
- S, M, R are categories, with internal pullback structure, giving recursive pullbacks

Recursive Pullbacks

A node of a
pullback may
itself
be a pullback

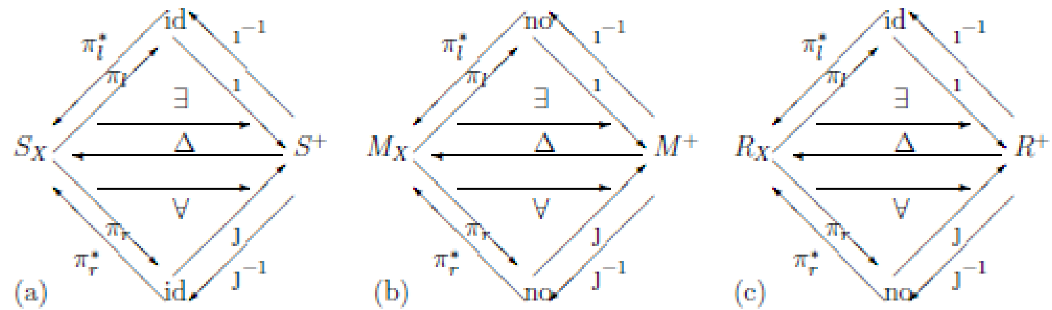
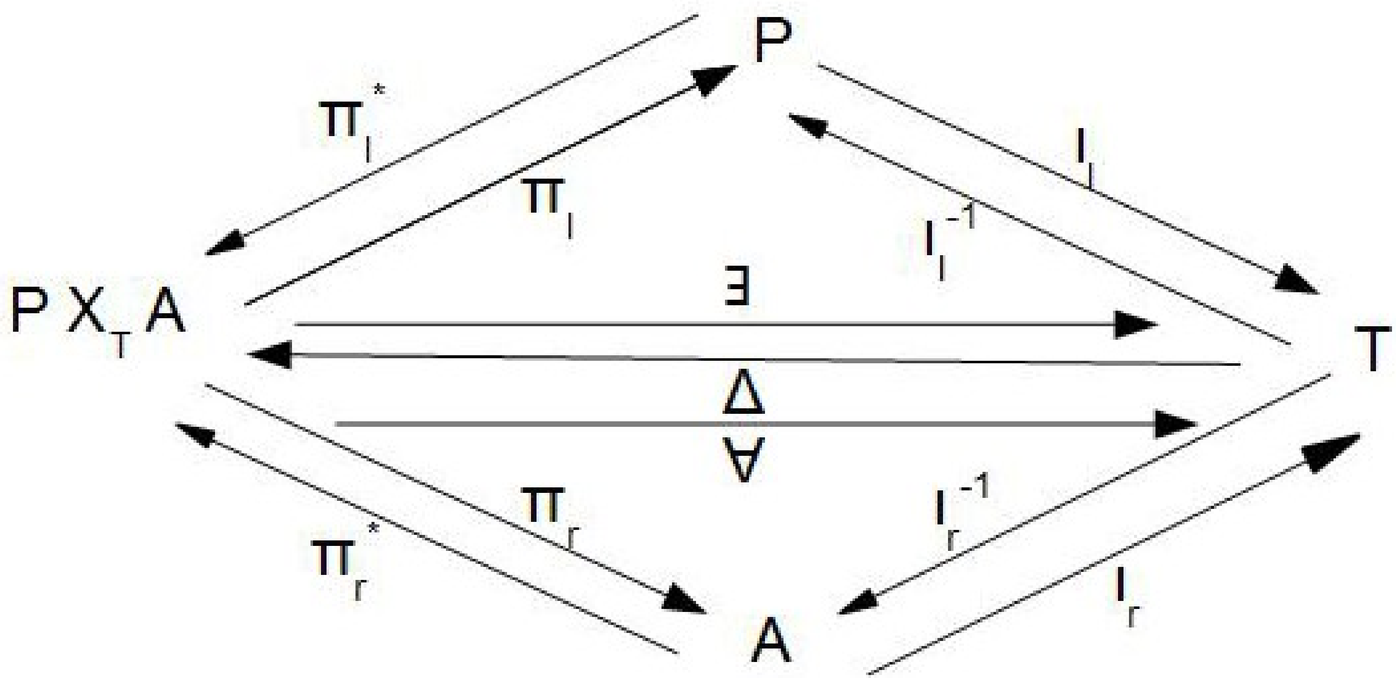


Figure 13: Internal Structure of Categories: a) The Pullback in **S**. S_X is id $\times_{S^+}$ id, S^+ is name $+_{\mathrm{id}}$ address. b) The Pullback in **M**. M_X is no $\times_{M^+}$ no, M^+ is title $+_{\mathrm{no}}$ grade, c) The Pullback in **R**. R_X is id $\times_{O^+}$ no, R^+ is mark $+_{\mathrm{id+no}}$ decision.

Each node in the pullback for Student over Marks in context of Result
is itself a pullback, giving a recursive structure

Pullback - Single Relationship: Bank Transactions by Account and Procedure



Pullback - Single Relationship Details

- $P \; X_T \; A$ (Procedure $X_{Transaction}$ Account)
  - Procedure is type of transaction: e.g. standing order, direct debit, ATM cash withdrawal
  - Account can belong to many users
  - Transaction is item for transfer of funds according to ACID requirements
- P, A, T are categories, with internal pullback structure, giving recursive pullbacks

Pullback - Two Pasted Relationships: Bank Transactions bv User/Account

Three Pullbacks Pb1, Pb2, Pb2 X Pb1

$$\begin{tikzcd}
(P \times_T A) \times_A U \arrow[rr, "\pi'_r"] \arrow[d, "\pi'_l"'] & & U \arrow[d, "\iota'_r"] \\
(P \times_T A) \arrow[rr, "\iota'_l", "\pi_r"'] \arrow[d, "\pi_l"'] \arrow[urr, phantom, "\text{Pb2}"] & & A \arrow[d, "\iota_r"] \\
P \arrow[rr, "\iota_l"'] \arrow[urr, phantom, "\text{Pb1}"] & & T
\end{tikzcd}$$

U is user
A is account
T is transaction

Usually written in horizontal (landscape) form. Vertical layout enables deep nested structures to be represented more readily

$$\begin{tikzcd}
(P \times_T A) \times_A U \arrow[r, "\pi'_l"] \arrow[d, "\pi'_r"'] \arrow[dr, phantom, "\text{Pb2}"] & (P \times_T A) \arrow[r, "\pi_l"] \arrow[d, "\iota'_l"', "\pi_r"] \arrow[dr, phantom, "\text{Pb1}"] & P \arrow[d, "\iota_l"] \\
U \arrow[r, "\iota'_r"'] & A \arrow[r, "\iota_r"'] & T
\end{tikzcd}$$

Pasting condition for Pb2 X Pb1: $\iota'_l = \pi_r$ after Freyd's Pasting Lemma

For our purposes, a pasted pullback is only a valid pullback if all inner and outer diagrams are pullbacks

Pasting is associative (order of evaluation is immaterial) but not commutative (relationship A:B 1:N is not same as A:B N:1)

Pullback – x10 Natural Bank Account Transactions

$$
\begin{tikzcd}
(((P\ X_T\ A)\ X_A\ U)\ X_U\ B)\ X_B\ C \arrow[r, "I_l''''"] \arrow[d, "\pi_l'''"'] \arrow[dr, phantom, "Pb4"] & C \arrow[d, "I_r'''"] \\
((P\ X_T\ A)\ X_A\ U)\ X_U\ B \arrow[r, "I_l'''", "\pi_r''"'] \arrow[d, "\pi_l''"'] \arrow[dr, phantom, "Pb3"] & B \arrow[d, "I_r''"] \\
(P\ X_T\ A)\ X_A\ U \arrow[r, "I_l''", "\pi_r'"'] \arrow[d, "\pi_l'"'] \arrow[dr, phantom, "Pb2"] & U \arrow[d, "I_r'"] \\
(P\ X_T\ A) \arrow[r, "I_l'", "\pi_r"'] \arrow[d, "\pi_l"'] \arrow[dr, phantom, "Pb1"] & A \arrow[d, "I_r"] \\
P \arrow[r, "I_l"'] & T
\end{tikzcd}
$$

$$
\begin{tikzcd}
C \arrow[d] \\
B \arrow[d, leftrightarrow] \\
U \arrow[d, leftrightarrow] \\
A\ X\ P \arrow[d, leftrightarrow] \\
T
\end{tikzcd}
$$

N:M and 1:N are handled by same pullback structure

$\downarrow$ 1:N $\quad$ $\updownarrow$ N:M

C company, B branch, U user, A account, P procedure, T transaction

10 pullbacks: Pb1, Pb2, Pb3, Pb4
Pb2 X Pb1, Pb3 X Pb2, Pb4 X Pb3
Pb3 X Pb2 X Pb1, Pb4 X Pb3 X Pb2
Pb4 X Pb3 X Pb2 X Pb1

For our purposes, a pasted pullback is only a valid pullback if all inner and outer diagrams are pullbacks

Invalid Pullback

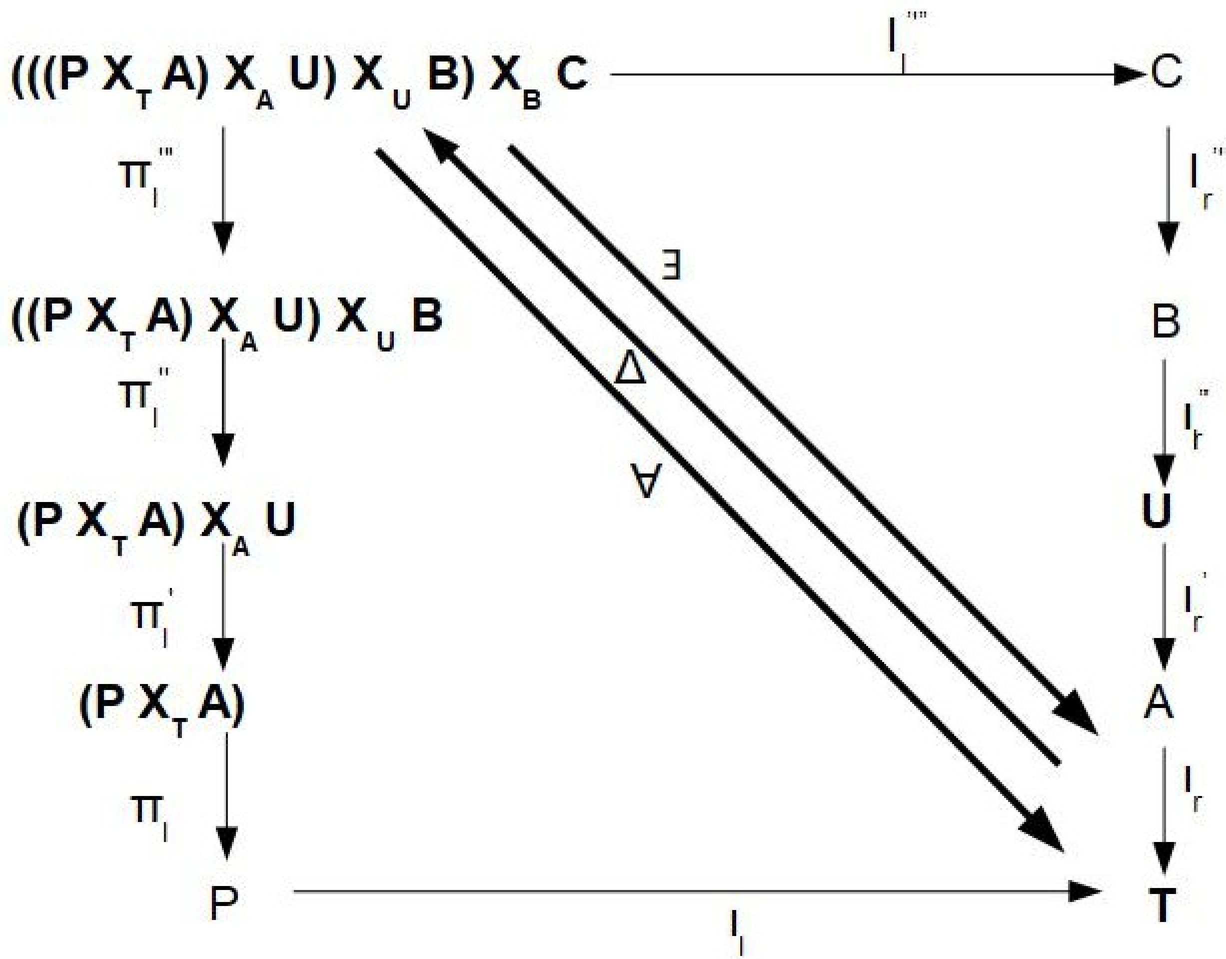
Invalid as
not all squares
are pullbacks

For instance
Pb4 X Pb2 is not
a pullback

$$
\begin{tikzcd}
((P\ X_T\ A)\ X_A\ B)\ X_B\ C \arrow[rr, "\pi_r''"'] \arrow[d, "\pi_l''"'] \arrow[rrd, phantom, "Pb3"] & & C \arrow[d, "I_r''"] & \\
(P\ X_T\ A)\ X_A\ B \arrow[rr, "I_l''", "\pi_r'"'] \arrow[d, "\pi_l'"'] \arrow[rrd, phantom, "Pb2"] & & B \arrow[d, "I_r'"', "I_r'''"] \arrow[rd, phantom, "Pb4"] & (B\ X_A\ U) \arrow[l, "\pi_l'''"'] \arrow[d, "\pi_r'''"] \\
(P\ X_T\ A) \arrow[rr, "I_l'", "\pi_r"'] \arrow[d, "\pi_l"'] \arrow[rrd, phantom, "Pb1"] & & A \arrow[d, "I_r"] & U \arrow[l, "I_l'''"] \\
P \arrow[rr, "I_l"'] & & T &
\end{tikzcd}
$$

Adjointness Holds for all Pullbacks

∃ ⊣ Δ ⊣ ∀ for this outer pullback and all other 9 inner pullbacks



Pasting Pullbacks – Discussion 1

- All diagrams commute
- All diagrams, inner or outer, are pullbacks
  - In pure maths, the condition is relaxed a little
    - Not appropriate for applied
- Structure is recursive
  - A pullback node may be a pullback structure in its own right
  - No limit to recursion

Pasting Pullbacks – Discussion 2

- Meets all Information System requirements as in 5NF (Project-Join)
  - Pullback is project-join through Π and Δ
  - Not categorification of set-theoretic approach
  - 5NF was a belated move by set-theoretic adherents to find a viable approach, which happened to follow category theory principles
- Pasting condition appears to be:
  - $\iota_l' = \pi_r$ (left-inclusion of outer square = right-projection of inner square)
  - Discussed further later

Pasting Pullbacks – Discussion 3

- Pasted structure
  - is a Cartesian Closed Category (CCC) with products, terminal object and exponentials
  - is a topos as a CCC with subobject classifier and internal Heyting Logic
- The subobject classifier provides an internal query language

The Pasting Condition 1

- $\iota_l' = \pi_r$ (left-inclusion of outer square = right-projection of inner square
  - Looks rather set theoretic
- But any pullback can be represented as an equalizer (ncatlab)

Equalizer for Pullback

$$P \times_T A \longrightarrow P \times A \underset{l_r \pi_r}{\overset{l_l \pi_l}{\rightrightarrows}} T$$

Maps relation onto product onto context via 2 paths through pullback

The Pasting Condition 2

Similarly for a pasted pullback, the equaliser is

$$(P \times_T A) \times_A U \longrightarrow (P \times_T A) \times U \overset{l_l \pi_l \pi_l'}{\underset{l_r \pi_r \pi_r'}{\rightrightarrows}} A$$

Equals in sets is undefined as context is not defined

Equaliser in categories, as a limit, is fully defined up to natural isomorphism

Subobject Classifier – Searching within a Topos - Boolean example

$$
\begin{tikzcd}
U \arrow[r] \arrow[d, "j"'] & 1_{topos} \arrow[d, "true"] \\
S \arrow[r, "X_j"'] & \Omega\ \{0,1\}
\end{tikzcd}
$$

Simple database query in category theory style

Ω {0,1} is subobject classifier; subobjects classified as either 0 or 1

X_j characteristic function is query mapping from object S to {0,1}, false or true

1_{topos} is terminal object of topos (handle on topos)

j is mapping from subtopos U (result of query) to object S

Diagram is actually a pullback of *true* along $X_{j.}$

U is $1_{topos}\ X_{\Omega\{0,1\}}\ S$

U is the identity of the subtopos, giving query closure

External Process

- Metaphysics (Whitehead)
- Transaction (universe, information system)
- Activity
  - Can be very complex but the whole is viewed as atomic – binary outcome – succeed or fail
  - Before and after states must be consistent in terms of rules
  - Intermediate results are not revealed to others
  - Results persist after end

Transaction is standard way of defining a Process

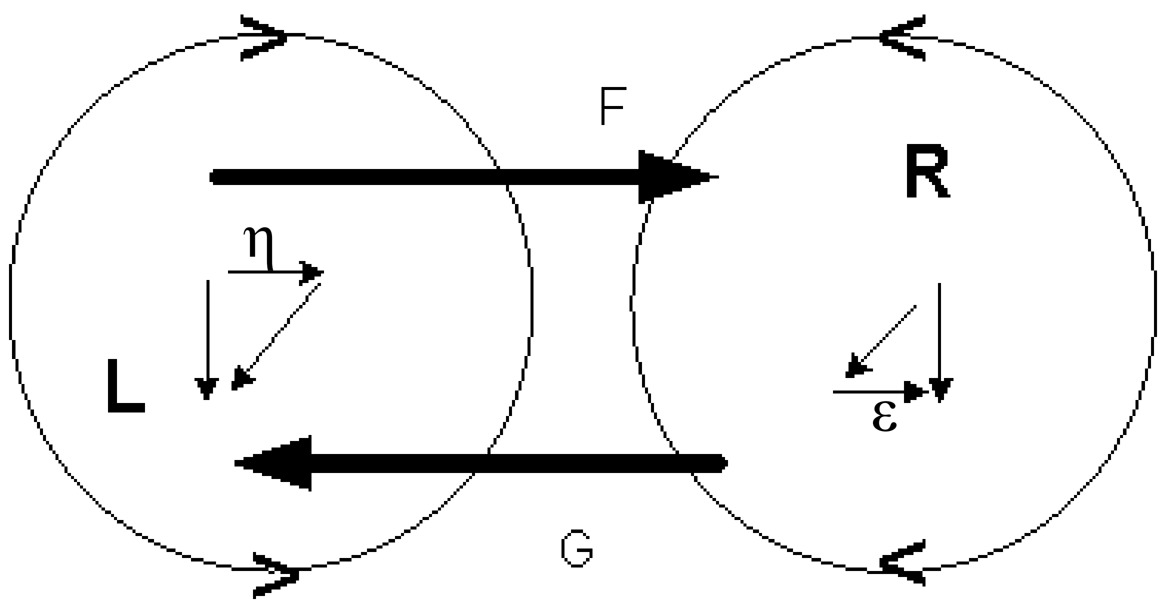
- Principles of ACID
  - Atomicity, Consistency, Isolation, Durability
- Logical technique for controlling the physical world
  - e.g. banking transaction
- Requires three cycles of adjointness between initial and target state
- First two for atomicity, consistency and isolation
  - First makes changes; second reviews changes
- Third for durability

Transaction in Category Theory

- In earlier work (ANPA 2010) we used adjointness to represent a transaction
  - Employing multiple cycles to capture ACID
- The aim now is to abstract this work using the monad, which we earlier described as the way forward
- The monad is an extension of the monoid to multiple levels
  - Monoid: $M \times M \rightarrow M$, $1 \rightarrow M$ (binary multiplication, unit)

Multiple 'Cycles' to represent adjointness

- Three ‘cycles’ GFGFGF:
  - Assessing unit η in L and counit ε in R to ensure overall consistency
  - 'Cycles' are performed simultaneously (a snap, not each cycle in turn)



F -| G

$\eta: 1_L \rightarrow GF(L)$ $\varepsilon: FG(R) \rightarrow 1_R$

Promising Technique - Monad

- The monad is used in pure mathematics for representing process
  - Has 3 'cycles' of iteration to give consistency
- The monad is also used in functional programming to formulate the process in an abstract data-type
  - In the Haskell language the monad is a first-class construction
    - Haskell B. Curry transformed functions through currying in the λ-calculus
    - The Blockchain transaction system for Bitcoin and more recently other finance houses uses monads via Haskell
      - Reason quoted is it's a simple and clean technique

Monad/Comonad Overview

- Functionality:
  - Monad
    - $T^3 \rightarrow T^2 \rightarrow T$ (multiplication)
    - 3 'cycles' of T, looking back
  - Comonad (dual of monad)
    - $S \rightarrow S^2 \rightarrow S^3$ (comultiplication)
    - 3 'cycles' of S, looking forward
- Objects:
  - An endofunctor on a category X

Using the Monad Approach

- A monad is a 4-cell <1,2,3,4>
  - 1 is a category X
  - 2 is an endofunctor ($T: X \to X$, functor with same source and target)
  - 3 is the unit of adjunction $\eta: 1_X \to T$ (change, looking forward)
  - 4 is the multiplication $\mu: T \times T \to T$ (change, looking back)
- A monad is therefore $\langle X, T, \eta, \mu \rangle$

The Monad as a 'triple'

- A monad is sometimes called a triple as by Barr & Wells. Term disliked by some as too set theoretic, e.g. Mac Lane
- Why a triple when 4 terms above?
- <X, T, η, μ> is reduced to
  - < T, η, μ> as category X is implicit in T
- True to the spirit of category theory a monad works over 3 levels, as 3 levels gives naturality
- So the laws we are going to see involve T (endofunctor performed once), T^2 (performed twice) and T^3 (performed 3 times)
- Note this multiple performance matches our transaction approach, outlined earlier with GF performed 3 times

The Comonad

- The dual of the monad
- A comonad is a 4-cell <1,2,3,4>
  - 1 is a category X
  - 2 is an endofunctor ($S: X \rightarrow X$, functor with same source and target, S is dual of T)
  - 3 is the counit of adjunction $\varepsilon: S \rightarrow 1_X$ (change, looking back)
  - 4 is the comultiplication $\delta: S \rightarrow S\ X\ S$ (change, looking forward)
- A comonad is therefore $<X, S, \varepsilon, \delta>$ or $<S, \varepsilon, \delta>$

Laws for the Monad

- Book-keeping
- Associative Law
- Unit Law

Associative Law for Monad

- The laws involve T^3 (3 'cycles') with the Associative law:

$$
\begin{tikzcd}
T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}
\quad \text{(a)}
\qquad\qquad
\begin{tikzcd}
S \arrow[r, "\delta"] \arrow[d, "\delta"'] & S^2 \arrow[d, "S\delta"] \\
S^2 \arrow[r, "\delta S"'] & S^3
\end{tikzcd}
\quad \text{(b)}
$$

a) Associative law for monad <T,η,μ >; b) Associative law for comonad <S,ε,δ>

Unitary Law for Monads

- The diagram commutes

$$
\begin{tikzcd}
IT \arrow[r, "\eta T"] \arrow[d, "="'] & T^2 \arrow[d, "\mu"] & TI \arrow[l, "T\eta"'] \arrow[d, "="] \\
T \arrow[r, "="'] & T & T \arrow[l, "="]
\end{tikzcd}
$$

Figure 13: Left and Right Unitary Laws for Monad $T =< T, \eta, \mu >$

Monad can be based on an adjunction

- The transaction involves GF, a pair of adjoint functors F -| G
  - F: X → Y
  - G: Y → X
- GF is an endofunctor as category X is both source and target
- So T is GF (for monad)
- And S is FG (for comonad)

3-cell descriptors with adjoints

- The 3-cell monad < T, η, μ>
  - is written <GF, η, GεF> (last up a level for multiplication)
- The 3-cell comonad <S, ε, δ>
  - is written <FG, ε, FηG> (last up a level for comultiplication)

Terminology

- A monad is often simply addressed by its endofunctor.
  - So $< T, \eta, \mu>$ is called the monad T
- Similarly for the comonad
  - $<S, \varepsilon, \delta>$ is called the comonad S
- It's a synecdoche

Operating on a Topos

- The operation is simple:
  - $T: E \rightarrow E'$
    - where T is the monad $<GF, \eta, G\varepsilon F>$ in E, the topos, with input and output types the same
- The extension (data values) will vary but the intension (definition of type) remains the same
- Closure is achieved as the type is preserved

The T-algebras – Changing the Definition

- More fundamental change to the operand (X or E)
- Produces a new consistent state of adjunction with modified intension
- The T-algebras manipulate the category X, when defined within a monad T
- They were developed in work by Eilenberg & Moore published in 1965

T-algebra defined

- For a category X, not necessarily a topos, in the monad <X, T, η, μ>, the effect is to obtain:
  - $\langle G^T F^T, \eta^T, G^T \varepsilon^T F^T \rangle : X \rightarrow X^T$
- That is a new monad adjunction $F^T \dashv G^T$ is defined to accommodate the changed category X^T
- For a topos E, this is equivalent to a change to E^T
  - $\langle G^T F^T, \eta^T, G^T \varepsilon^T F^T \rangle : E \rightarrow E^T$

The T-algebra

- For a monad $\langle T, \eta, \mu \rangle$ in X
- A T-algebra is:
  - $\langle x, h \rangle$
- Where x is an object in X
- And $h: Tx \to x$ is the structure map of the algebra
- Such that the following diagrams commute

T-algebra: Associative/Unitary Laws

Associative

$$
\begin{tikzcd}
T^2x \arrow[r, "Th"] \arrow[d, "\mu_x"'] & Tx \arrow[d, "h"] \\
Tx \arrow[r, "h"'] & x
\end{tikzcd}
$$

Unitary

$$
\begin{tikzcd}
x \arrow[r, "\eta_x"] \arrow[dr, "1"'] & Tx \arrow[d, "h"] \\
& x
\end{tikzcd}
$$

Both diagrams must commute for T to be a monad

Other Monadic features

- Kleisli Category of a Monad
  - Transforms a monad into a form more suitable for implementation in a functional language
    - Used in Haskell rather than the pure mathematics form of Mac Lane
- Beck's Theorem
  - Provides rules on which categorial transformations in the T-algebra $X \rightarrow X^T$ are valid.
    - Sometimes called PTT (Precise Tripleability Theorem)

Cartesian Monads

- If underlying categories are pullbacks
  - AND T preserves pullbacks
  - AND μ and η are Cartesian
- Then the monad is Cartesian
  - Facilitates its use in transformations where a Cartesian type is expected

Summary of Progress

- Topos has been established as data-type of choice
- Monad shows potential for processing the topos and for transforming the topos

- Areas for attention:
  - Intension/extension in topos, including pullbacks, subobject classifier and operations by the monad
  - Exploring usefulness of additional work on monads including those mentioned here: T-algebra, Kleisli, Beck and Cartesian monad