FORMAL REPRESENTATION OF "PROCESS & REALITY" IN THE METAPHYSICAL LANGUAGE OF CATEGORY THEORY: WHITEHEAD'S RELATIONAL THEORY OF SPACE

MICHAEL HEATHER & NICK ROSSITER

Department of Computer Science and Digital Technologies
Northumbria University, NE1 8ST, UK

michael.heather@trinity.cantab.net; nick.rossiter1@btinternet.com
http://www.nickrossiter.org/process/

Abstract: Category theory was not sufficiently developed in his lifetime for Whitehead to apply to his speculative metaphysics of *Process & Reality*. The natural (assumption-free) topos as a cartesian closed category is now able to conceptualise formally the inherent space-time structure of Whitehead's extensional space that he appreciated is beyond the metrics of the classical mathematics he helped to develop. This paper examines the background to a possible formal representation using Category Theory for Whitehead's metaphysical cosmology starting with his notion of extensional space which is beyond finitary mathematics.

1 BACKGROUND

Alfred North Whitehead was one of few to appreciate 'the theory of linear extension'[1] [21] of the eminent philologist Hermann Grassmann (1809-1877). 'Unfortunately, when it was published', observes Whitehead[2], 'nobody understood it; he was a century ahead of his time.'. Nevertheless that publication was to lead to linear algebra, vector spaces, differential geometry and the mathematics that underpinned much of 20th century science ([42] at pages 203 *et seq*). Grassmann's studies contained a germ of category theory to be pursued here for in the meantime it has led to mathematical topics like universal algebra, topology and homotopy all of which are subsumed in category theory.

1 *Die Lineale Ausdehnungslehre, ein neuer Zweig der Mathematik*
2 [64] Essays in Science and Philosophy, Part I, Personal, Autographical Notes at p 12

In particular Grassmann's insight[3] allowed geometry to escape from the metric dimensions of Euclid which is the essence of Aristotle's natural 'assumption-free' *topos* in metaphysical category theory that is the subject of this paper. That now well describes the connectedness that Whitehead outlines for the structure of his 'cosmology' but which he was unable to represent formally even as a world-class mathematician of his age. It is a long road to represent Whitehead's *Process & Reality* formally but until it can be studied in this way his speculative metaphysics cannot expect the impact it deserves. The next step will be to represent formally the entity types of *process* that populate his *reality* but there is no space to attempt that here.

Whitehead himself pursued Grassmann's ideas with his own *Universal Algebra* [52] but his early career may be characterized as a somewhat frustrated author of mathematical texts. *Universal Algebra* led to his election as a fellow of the Royal Society of London but he was disappointed to find that the work was not widely accepted and understood. His projected second volume was therefore abandoned in favour of a joint treatise with his student Bertrand Russell on the logical basis of mathematics. He and Russell attended the renowned Paris 1900 International Congress of Mathematicians in Paris and felt further inspired by interaction with the likes of Hilbert, Frege and Peano. The outcome was the well known but little read *Principia Mathematica*.

The work was probably as frustrating to write as it is to read. Part II (at pp 328-383) of the first volume attempts to define the cardinal numbers 1 and 2 but without success. Volume II on the other hand devotes 724 pages in an unsuccessful attempt to formalize the arithmetic axioms of Peano and fails even to establish the fundamental $1 + 1 = 2$. The explanation for all this we now know with hindsight is because the natural topos lacks a natural number object[4]. Whitehead's dismay and disappointment with the whole project of the *Principia Mathematica* is understandable. Not surprisingly the fourth volume on geometry was never published even though apparently much of it

3 As preeminently a philologist insight that may very well have come from Grassmann's studies in natural language.

4 The natural number object can only be introduced by assuming a successor function requiring an axiom of choice to import some closed world assumption.

had been written. Indeed Whitehead did not involve himself with the material in the second edition of 1925-1927 at all, other than to proof-read drafts[5]. Nevertheless it appears that in the meantime Whitehead was diverting to a physics approach to geometry through his Harvard Lectures [8] that emerges in *Process & Reality* [61] The fourth volume was therefore turning out to be a nightmare, based on some conflicting transitional state. Russell alludes to this in the preface of [49] when commenting on the problem of scientific method in philosophy:

> I have been made aware of the importance of this problem by my friend and collaborator Dr. Whitehead, to whom are due almost all the differences between the views advocated here and those suggested in *The Problems of Philosophy*. I owe to him the definition of points, the suggestion for the treatment of instants and "things", and the whole conception of the world of physics as a construction rather than an inference. What is said on these topics here is, in fact, a rough preliminary account of the more precise results which he is giving in the fourth volume of our *Principia Mathematica*. ([49] preface p 8).

We are concerned here with the subject matter of the fourth volume so far as it relates to Whitehead's concept of extensional space but not from the viewpoint of the history and sociology of science. Readers interested in that perspective are referred to the studies of Patrick J Hurley[6] who cites Whitehead's displeasure at Russell's disclosure of the fourth volume material in [49] expressed in the letter to Russell:

> I am awfully sorry, but you do not seem to appreciate my point. I don't want my ideas propagated at present either under my name or anybody else's – that is to say, as far as they are at present on paper. The result will be an incomplete misleading exposition which will inevitably

5 The Introduction to the Second Edition begins with the wording 'In preparing this new edition of Principia Mathematica, the authors have thought it best to leave the text unchanged' ([54] at p xiii) but the second edition has spawned quite a literature of its own because of the way it developed the logic itself. See [37] for further details.

6 See www.religion-online.org/showarticle.asp?title=2469

> queer the pitch for the final exposition when I want to put it out. My ideas and methods grow in a different way to yours and the period of incubation is long and the result attains its intelligible form in the final stage, – I do not want you to have my notes which in chapters are lucid, to precipitate them into what I should consider as a series of half truths . . . ([50] at p.78).

Whitehead's other writings of the ensuing 'period of incubation' [53, 55, 57, 58, 59, 60, 62, 63] suggest that Whitehead was for some time confident that his extensional theory of space presented in Paris in 1914 [49][7] and consisting of material intended for the fourth volume could still be expressed mathematically as distinct from Russell's 'half truths'. Whitehead's alternative 1922 theory of relativity [59] probably marks the watershed after which he realized that it is a relativistic quantum world we inhabit beyond classical mathematics. A lesser mathematician might have persevered with the tensor mathematics but he clearly appreciated that his earlier logicalism was inadequate to articulate his speculative metaphysics. Indeed Gödel has confirmed this in his doctoral thesis ([19] at p2) by expressly using the axioms of Whitehead's *Principia Mathematica* as the basis to prove that first order predicate logic is complete but only for closed systems. Metaphysics on the other hand is of higher order but amenable to the intuitionistic internal language of the natural topos derived from physics and therefore outside of Gödel's theorems.

Of course this strand from Grassmann was only one of many influences on Whitehead as comes through from the text of *Process & Reality*: Plato, Aristotle, Leibniz, Newton, Kant, Locke, Hume, etc; but beside the general proposition that science progresses better when supported by mathematics there seems little from them of direct relevance to formalizing the geometry of space. There were also contemporaries like Poincaré, James, Bergson, Dewey, Husserl, Einstein, Bohr, etc, who get little or no mention in *Process & Reality* but who nevertheless were providing a climate of thought operating heavily

7 There is apparently no original English version extant but details of its publication in French with its subsequent translations into English and commentary may be found at the religion-online website given in the footnote above.

on Whitehead's mind that it was necessary to escape the limitations of classical mathematics – but again these seem of little direct relevance here. What at first sight does appear more relevant, but in the foreground rather than background, is the content of Part IV of *Process & Reality* itself and the work of those who have since sought to build on it.

2 FOREGROUND

The extraneous evidence outlined above suggests Part IV of *Process & Reality* entitled simply *The Theory of Extension* is the material written much earlier as proposed contents for the projected fourth volume of *Principia Mathematica* on geometry that was never published. Certainly it appears as an insert differing markedly from the rest of *Process & Reality* as the only part in any way mathematical. It is geometric in tone but in a very idiosyncratic style reminiscent more of Venn diagrams[8] than Euclidean geometry. On the one hand it does not adhere to the strict logical principles adopted in the first volume of *Principia Mathematica*. It makes assumptions that are beyond self evident primitives and lists definitions that are more than mere labels, as banned in the introduction of the first volume. On the other hand it is clear from internal evidence of the text that it is much more than an opportunity to get published material already written for another occasion. It incorporates more recent published work of others. For example definitions of Professor T de Laguna ([65] at pp 287, 295, 297) are fundamental to the main thrust of Part 4 and indicate that the whole subject of the extensional theory of space had been recast in Whitehead's mind. This is also further evidence (as from his letter to Russell cited above) that he was struggling perhaps for nearly twenty years with a formal description for space. Part IV is the then current version of Whitehead still trying his hand at representing connectivity in the reality extension of his world of process.

Some of his observations are very pertinent here. Thus the overlap in the diagrams he makes into 'ovate classes'. This is a perspective of

8 in a couple of pages ([65] pages 295 *et seq*)

universal limits in category theory the significance of which was not appreciated until the 1970s. The impossibility of producing adequate diagrams[9] to represent such features also add weight to the proposition that we need to turn from mathematics to physics for nature produces an abundance of limits and co-limits, indeed everywhere all the time. However the mathematics of Part IV does not really go anywhere and Whitehead did not take it any further in the remaining twenty years of his life.

However the disciples of genius often with great enthusiasm attempt to take the work way beyond where the master would have gone and subsequent events show that the subject of Whitehead's connectivity has many potential onward paths to pursue. Geometrical connectivity is one aspect of atomicity and raises issues of whole-part relationships. Whitehead has in this context spawned interest in some new disciplines like holism, point free geometry, mereology, and mereotopology. These have generated a considerable literature[10] and attempts to define new formal systems of logic. Examples of these are the work of [10], [51] and [12] but the latter have demonstrated that the formal representation can be reduced to Boolean systems. Boolean logic however is not inherently constructive and does not have the intuitionistic logic required by physics. Although the subject matter of that field of work is within the ambit of this paper they will not therefore be examined in detail here as the end result is a 'null return'. These are now mainly of only historical interest.

Although Einstein's relativity and quantum theory were contemporary with his 'period of incubation' and a clear catalyst for *Process & Relativity* Whitehead made no serious attempt to include any quantum mechanics in his theories. Perhaps his brief abortive incursion into the subject of relativity [59] dissuaded him. This century Michael Epperson has made an 'attempted correlation of quantum mechanics and Whitehead's cosmological scheme' [15]. That attempt has taken the form of a painstaking recasting of Whitehead's metaphysical cate-

9 The diagrams of transition functions between overlap of manifold patches in twistor cohomology, for example Figure 33.17 in [42] at p 988, are perhaps more advanced developments of ovate classes in Whitehead's diagrams

10 and vocabulary like *gunk* for any whole with proper parts.

gories in a Hilbert space with Dirac notation. This seems rather to miss the point that a Hilbert space is composed of points which are just numbers, even though rather sophisticated numbers and are echoes from Whitehead's discarded former life of *Principia Mathematica*.

More recently Epperson has published *A Topological Approach to Quantum Mechanics and the Philosophy of Nature* [16]. 'Topology' refers to the use of a sheaf theory approach from co-homology. The publishers had issued pre-publicity with the title *Foundations of Relational Realism: Quantum Mechanics, Category Theory, and the Philosophy of Alfred North Whitehead*. The change from 'category theory' to 'topological approach' perhaps suggests that some original aim to use category theory was not realised. The change is quite significant because it undermines the decoherence thrust of the book. Metaphysical category theory can track the space-time development of the quantum wave function. Application of topology immediately collapses it. Sheaf theory is a finitary description of the *pullback* in full category theory. We have already shown in 2002 how this provides a simpler but more sophisticated approach to quantum theory [24, 25]. The significance of quantum decoherence within process which as briefly discussed below is more simply represented as monadic composition in the natural topos.

3 THE SIGNIFICANCE OF CATEGORY THEORY

Around the time of Whitehead's death in the 1940s formal 'category theory' emerged to subsume algebra, geometry and topology as a formal metaphysical language that is now able to integrate his natural philosophy and mathematics to culminate in what we might explore here as the implied formal ingredients of *Process & Reality* as it climbs up through two levels from models to metaphysics. A model reduces reality that metaphysics generalizes.

Just as a mathematical theory is an instantiation of the world so the world is an instantiation of metaphysics. For historical reasons category theory has had to develop from within classical mathematics and current text books deal mainly with the category of sets that resides within the discarded mathematics of Whitehead's early period and therefore cannot deal adequately with his speculative metaphysics. For as metaphysics generalizes the dynamics of nature, metaphysical language relates to natural process without the need for the arbitrary axioms of mathematics. Fortunately therefore metaphysical category theory is simpler than the category theory of classical mathematics and also greatly simplifies the natural language descriptions that flowed from the pen of the author of *Process & Reality* that are difficult for those of us not endowed with the power of his mind. The formal categories are therefore simpler than the natural language expressions but it is a simplification satisfying his own observation that "the only simplicity to be trusted is the simplicity to be found on the far side of complexity."

Rather paradoxically mainstream science a century later is still trying to understand our world using the models based on the concepts of his early mathematical period rather than the informal categorical approach enunciated in the 1929 *Process & Reality* of his later philosophical period. The current mainstream position at the turn of the twenty-first century is probably well summed-up in Penrose's encyclopaedic tome entitled *The Road to Reality, A Complete Guide to the Laws of the Universe*:

> There have also been other intriguing radical proposals, such as those of Richard Jozsa and of Christopher Isham which employ topos theory. This is a kind of set theory arising from the formalization of 'intuitionistic logic' (see Note 2.6), according to which the validity of the method of 'proof by contradiction' (§2.6, §3.1) is denied! I shall not discuss any of these schemes here, and the interested reader is referred to the literature. Another idea that may someday find an significant role to play in physical theory is category theory and its generalization to n-category theory. The theory of categories, introduced in 1945 by Samuel Eilenberg and Saunders Mac Lane, is an extremely general algebraic formalism (or framework) based on very primitive (but confusing) abstract notions, originally stimulated by ideas of algebraic topology. (Its procedures are

> often colloquially referred to as 'abstract nonsense'.) ([42] at p.960)[11].

The typo 'an significant role' in this short extract suggests that Sir Roger was unhappy with this sentence and had not finished editing it. The reference to Jozsa is to an unpublished thesis he supervised and those to Isham relate to books he edited but none seem to give any adequate treatment of a topos or category theory. Although the book has in its title 'A complete guide to the Laws of the Universe' nevertheless on its own admission is confused by the notions of category theory. There is clearly a serious misconception on the significance of intuitionistic logic in constructive mathematics. The single exclamation mark about the validity of proof by contradiction raises a shadow over the whole thousand pages of the book and fuels our belief that any scientific theory today is suspect unless it can be validated by category theory. It follows that such validation also tests the correlation of any scientific theory within Whitehead's scheme of speculative metaphysics. It was probably Alexandre Grothendieck of the Bourbaki group who was the first to see the depth (or more accurately the 'heights') of significance in the topos. Aristotle was of course responsible for promoting the metaphorical connotations of the simple word for *place* in Ancient Greek and there is a parallel abstract usage to be found in a literary context. A major feature in Aristotle that cannot be captured by finitary mathematics is the macrocosm-microcosm relationship where the part has the characteristics of the whole. Whitehead alludes to this relationship in *Process & Reality* and seems to use the terms interchangeably with 'macroscopic and microscopic'. Finitary mathematics is unable to represent the relationship directly because a set cannot be a member of itself in axiomatic set theory and cannot be proved to be consistently defined in naive set theory. Likewise unfortunately the Grothendieck topos does not manage to escape from its Bourbakian roots in Hilbert's finitary methods[12] which also serves to make it unnecessarily complicated. The same over-complexity may be found in the standard category theory texts

11 the Note and § numbers refer to the book; 'n-category theory' is considered below.

12 Colin McLarty recently claims that 'the entire Grothendieck apparatus' is of weaker strength than finite order arithmetic [39].

of classical mathematics[13]. The approach from physics on the other hand by identifying the arrow of category theory with process in nature greatly simplifies the complexity and enables category theory to act as an Occam's razor and as a very powerful scientific tool. For a parallel view as an alternative natural philosophy, see [10].

4 PROCESS IN CATEGORY THEORY

Process is concerned on the one hand with the global/local distinction and on the other hand by the juxtapositioning between the stationary and the non-stationary. The universal and the particular, the static and the dynamic are integrable at the level of metaphysical reality and are formally representable and accessible in natural categories that follow physics. Process relates not just to the non-stationary but subsumes both the static and the dynamic. One is contained in the other but which way round? Such problems like Zeno's paradox of the arrow's dynamic flight consisting of only static positions are avoided in the 17th century French logic school of the Port Royal, following from Aristotle's first and second intentions, as distinguishing the intension from the extension. Information systems are an important example of the need to represent the world by computer. Within such systems, the limitations of modelling reality are conspicuous in database methods, as discussed later.

Natural categories as a metaphysics provide mixed levels for intension/extensions. Intension and extension alternate in a preorder, that is with arbitrary beginning. This is the natural role of the categoreal arrow with an identity arrow as intension and a distinguishable valued arrow for extension. The simplest identity arrow is treated as an object, the next higher identity arrow (the functor) composed of extensional arrows between objects makes a category with ordinary functors as extensional arrows between categories. The next higher level

13 The reason for this is that mainstream pure mathematics is still operating on the basis of Whitehead's earlier PM which is limited to first order theories. See the recent [9] as an example of lengthy descriptions needed for the more restrictive use of a model Topos.

is a category of these categories with objects as categories and functors between them. The highest level arrow is again an identity natural transformation which composes the previous level of categories as objects with natural transformations between them. It is this final identity natural transformation that constitutes a topos. The whole is just a recursive system with closure at four levels consisting of three open interfaces. The identity natural transformation is scale invariant and any higher level would only be a reformulation of this same level. This is process and the Universe is an instantiation of process but the World is even greater than the physical Universe consisting of all the relations between physical entities and all the relations between those relations. The interaction of subjective human behaviour in the global world of physics, biology and economics is very topical.

Relationships in nature are explicable in natural categories with the single concept of adjointness that consists only of a pair of contravariant arrows inducing a monad. Mathematical categories other than the Cartesian closed are possible but natural categories being derived from physics only recognise the existence of Cartesian closed categories where every object is the domain of a covariant arrow and the co-domain of a contravariant arrow. This recursive structure applies at any of three possible levels and in general occurs between a pair of categories where adjointness of the pair of arrows contravariant to one another between the categories induces a monad (or 'triple') and a co-monad (or 'co-triple'). Each arrow has a dual role. One is the contingent arrow of intension (in the monad) and the determinant arrow of extension (in the co-monad) while the other arrow is the contingent arrow of extension (in the co-monad) and the determinant arrow of intension (in the monad). Finitary mathematics seeks to model various features of the adjointness between natural categories in a range of topics, including currying, lambda calculus, Kan extensions, Galois theory, the Yoneda lemma and its embedding.

The fundamental structural significance in the world is the way the local connects into the global such as in the McLuhan Global Village where everything is connected [40]. The temporal analysis is the distinction between stationary and the non-stationary. Philosophically this global/local distinction is not at all new. It is at the root of Zeno's paradox of the arrow's dynamic flight consisting only of static positions.

The noun 'process' or the participle 'processing' commonly describe an act of transforming an existing object by some procedure to another form as in a manufacturing or business administration procedure. Wikipedia deals with its entry for Process in up to 40 different fields of knowledge, including philosophy, science, engineering, computing, chemistry, biology, law, business and even the 'process haircut'. There are variations in the meaning of the word depending on context. For instance in business, process describes activities or tasks that produce a specific service or product for customers. Interestingly Wikipedia does not include physics in its lists of fields for process.

The whole subject of cybernetics can be viewed as a process operating in nature as in Wiener's definition [66] involving comparison of communication in the animal and the machine [45]. Process describes the way that both animals (biological systems) and machines (non-biological or "artificial" systems) can operate according to cybernetic principles. This was an explicit recognition that both living and non-living systems can have purpose. Wiener considered that systems theory seeks to deal with the local/global divide [45], treating systems as equivalent to process but the latter is the higher form. The early specification of the working of the brain in cybernetics by Ashby [2] amounts to the concept of process but it was von Bertalanffy of the early founders of cybernetics that explicitly related the latter to process [6, 7]. Whitehead had some influence on philosophy of business administration and organizational theory, through developing a focus on identifying and investigating the effect of temporal events, as opposed to static things [28].

Most writers trace process to the 'all is flux' of Heraclites in contrast to Parmenides, who is more usually associated with a static view. However, process is more than flux and also subsumes permanence. It is rather the Heraclites' *logos* which was taken up by the Greeks of Alexandria and the Judeo-Christian tradition to identify *logos* with God and the second person in the Trinity. The whole theory of evolution is process too but one where the origin of species does not unfold in a linear fashion. Evolution appears a foundational natural process encompassing both emergence and change. Ordering is adjointness and includes both static and dynamic aspects. It is a paradox that process includes invariance[14] which describes no change under a transformation. Indeed scale invariance turns out to be an important phenomenon of process. Fractal patterns arising from scale invariant physics are studied piecemeal with use of special sets like the Mandelbrot, Julia etc. However general methods are restricted because a set cannot be a member of itself in the way that a reflective subcategory can have itself as an object. Information systems like the web also exhibit properties of scale invariance but we do not have space here to pursue this aspect of process which arises in exponential categories.

There is always the problem of where to begin. That statement may be formally expressed as *a pre-order of categories* or just as well as *a category of preorders* for both lack beginning and ending. However within process we can but focus on the *category of reality* in the sense of the category where objects and relationships between objects exist to make up the physical world. This is metaphysical process and the Universe is an instantiation of process but the World is even greater than the physical Universe consisting of all the relations between physical entities and all the relations between those relations. Physical relations connect directly from higher-order relations. This is treated bottom-up but because of the holistic nature of process it is driven top-down. A topical example is the recent realisation of how subjective human behaviour affects the objective syntax of world economies. Current practical examples of applied recursion across levels is deduplication in structured data storage [32] or functional DNA nanostructures that can be integrated into larger structures as miniature circuit boards in bioengineering [47].

In this sense the *World* is greater than the physical Universe of cosmology. There is a unique arrow from the source of the World to every object in it and a unique resultant arrow between any pair of objects as in Figure 7.1.

For we are concerned with the higher order of relations between physical objects and the relations between those relations which to-

14 The subject of invariance was mainly developed in the 19th century by Arthur Cayley. Saunders Mac Lane [41] traces the early origins of category theory to Cayley.

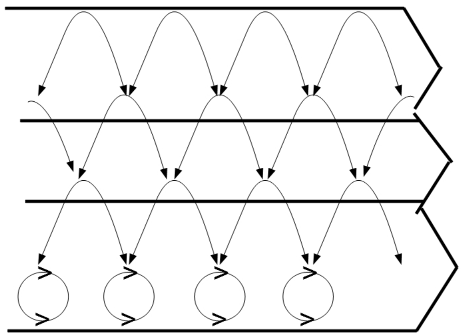

Figure 7.1: A Schematic World-Universe Relationship. The lowest horizontal arrow is a category consisting of a row of notional elementary objects connected by a row of vertical arrows which are themselves connected by a higher row which are in turn connected by a yet higher row.

gether with the physical objects of the Universe make up the World. This then embraces the whole of human affairs and activity including the arbitrary disciplines of philosophy and theology. Existence in categories is identifiable with the object which as we shall see is the condition known categorically as 'Cartesian'. Ordering is adjointness and includes both static and dynamic aspects.

This empirical knowledge that 'every entity that exists is related to every other object that exists' is no more than a definition of the Universe to include everything naturally accessible. This provides a unique direct arrow between any pair of objects that is the composition of all possible arrows between them. This is the structure given the label preorder. Figure 7.3 presents a two-dimensional representation for the context of the objects C and A of a preorder. There is but one unique arrow between any pair of objects in a preorder and that arrow as the figure shows is the limit of all other possible arrows whether directly between the pair or indirectly via any other pair in the preorder. We cannot assume any orientation for it or even presume the concept of a dimension. It is possibly easier to imagine than to draw, although our common perception wants us to imagine it in

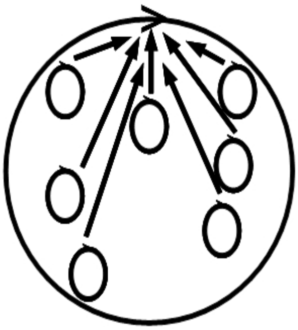

Figure 7.2: Terminal Object. A category with a terminal object has an arrow from every object to it. In preorders this arrow is unique as in Figure 7.3

three dimensions or possibly in higher order algebraic or geometric dimensions but lies easily in higher-order geometric dimensions. A process preorder does not exist in any space whether algebraic or geometric. Rather it should be space free. This is the quantum world. However the effect between entities is mutual and the arrow is therefore two-way but not symmetrical because the opposing directions give rise to a natural parity in their mutuality. This is the ultimate reality of the quantum world. Whether it is the quantum or the physical world that is true, reality seems just a matter of personal preference.

Newtonian physics treated the universe as some container either rectilinear or spherical but embedded in time. Such a structure is representable, for example by Yoneda or Curry techniques[15], to first-order as a number. This is the classical model which can be verified by measurement in first-order predicate logic because as Gödel has shown first-order predicate logic is complete for a closed world. However Gödel has also shown that such a logic is not complete for an open world and any model based on number and relying on axioms is not complete whether open or closed [20]. This effectively sets a limit to

15 See [3] at pages 118 and 190 respectively

Wigner's 'unreasonable effectiveness of mathematics in the natural sciences' [67].

5 METAPHYSICS

If we want to identify a category with reality, existence requires designation of one object as the terminal object, as shown in Figure 7.2. This is the condition known as 'Cartesian'. It is also possible to designate another as the source of the process as initial object. This is the condition known as 'co-Cartesian' but is not a necessary and sufficient condition and may therefore result in over-specification and a too constrained system. There is a free functor mapping from the preorder on to any of its partial orders. It is natural to identify the terminal object with the covariant identity functor. If the initial object exists it would exist as the contravariant identity functor of the category. Nevertheless although these are arbitrary terms the use of the labels 'terminal' and 'initial' imports an interpretation and requires the existence of some axiom of choice, which is an axiom/assumption of set theory. Whitehead raises the stakes by ascending two levels from models to speculative metaphysics but nevertheless we can by the use of formal categories eliminate 'the speculative' and indeed any amount of assumption. Here we try to make no assumption at all beyond that the World exists. We try to keep open issues about terms such as 'terminal' and 'initial' because they may be related to what cosmologists currently tell us about the fabric of the physical Universe consisting mainly of dark matter and dark energy with only 4% in familiar forms.

Whitehead's *Process & Reality* leaps across current fundamentals at the frontiers rather than attempting incremental advances within existing knowledge. But how do the general and the particular relate within the structure of the world? Any formal description needs to be able to combine both the global and the local. This is possible with natural categories by substituting Whitehead's interpretation of metaphysical process in *Process and Reality* [61] for that in his earlier understanding in the *Principia* [54], which was the starting point for the

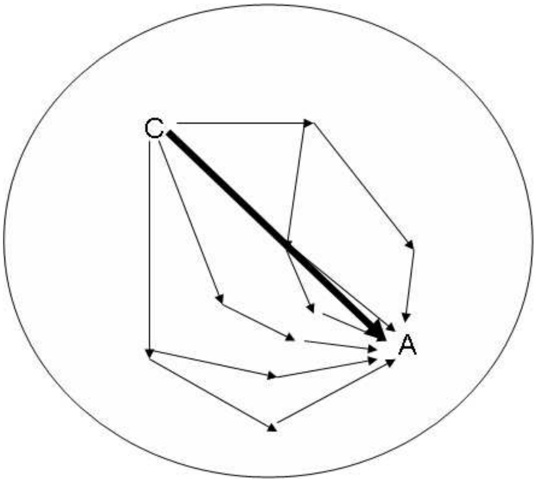

Figure 7.3: The unique arrow from C to A as a limit of other arrows in a unidirectional Preorder. The co-limit sums over all others

traditional finitary category theory of Eilenberg and Mac Lane [41]. It is the difference between a metaphysics and modelling which are separated by two levels as in Figure 7.4 (diagram 16 in [46]). Metaphysics is one level up from reality in human perception [17] while models are one level down. The limitations of modelling reality can be seen in information systems where there is a need to represent the world on computers. Problems are evident in database methods like ACID [22, 28] and in Codd's pure relational model [13]. In database design, data normalisation is used to attempt to match the logical data structures to the physical world. This method of design has a number of unsatisfactory features. Firstly it is difficult to enforce the laws of the physical world in the operational database and secondly the theoretical underpinning, based on set theory, is not natural because of the problem of representing arrows across multiple levels as functions.

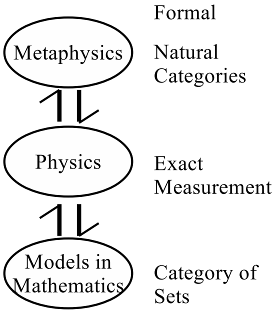

Figure 7.4: The Staircase of the World from Metaphysics to Models

6 FINITARY CATEGORIES MODEL NATURAL (METAPHYSICAL) CATEGORIES

Whitehead developed his theory *Process and Reality* in what he terms 'speculative metaphysical categories' These are in great contrast to the formal principles he enunciated with Bertrand Russell in *Principia Mathematica* and Whitehead devotes Chapter 1 of his later work ([65] 4-26) to explaining in a general philosophical context why they had to be speculative. For the second half of the last century has seen substantial advances in the development within finitary mathematics of formal categories, based on the concept of the arrow and initiated by Eilenberg and Mac Lane [41]. The phrase 'finitary mathematics' is a term first coined by the mathematician David Hilbert[16] and effectively describes the whole mainstream of twentieth century mathematics built up on a system of proofs in set theory and number from incompletely specified axioms. The adjective 'finitary' is itself a little misleading as finitary mathematics includes topics like infinity and transfinite numbers as these are modelled on the finite concept of number.

16 according to Feferman [20] Hilbert never defined finitary mathematics and it collapsed at its foundations under the weight of Gödel for the reasons mentioned above.

Nevertheless it is possible to ascend the staircase in Figure 7.4 from categories as mathematical models to metaphysical categories and extend that ladder to categories that are no longer speculative but which can now be made formal thanks to the work of Eilenberg, Mac Lane and a large number of pure mathematicians world-wide who have refined and extended their original interpretation of the humble arrow based only on the four properties:

1. a morphism from domain to co-domain
2. identity from an indistinguishable domain and co-domain
3. associativity
4. composition,

There are two distinctions important for process that we need to draw. One is between metaphysical categories and finitary categories in respect of the use of number in physics; the other is between sets and either types of categories in respect of the representation of intension and extension. We will first consider the natural numbers then look at intension and extension as an intrinsic property of parity to be found in adjointness.

7 THE FINITARY CATEGORY OF THE NATURAL NUMBER

Because it relies heavily on experiment, physics as a discipline has become identified with measurement and number as its prime conceptual tool. Consequently it has become very bound up with sets which equate to number. However it is an assumption that qualities and quantities are representable as number. The physics and the mathematics have become merged so that space is a complex number whether it is Newton's Universe as a container or the infinite Hilbert space of quantum mechanics. These it should not be forgotten are just numbers. This is fine to the extent of first order models for which Gödel (as mentioned above) has shown to be consistent but it is not sufficient for open or other higher order systems for Gödel has shown these to be neither consistent nor decidable when relying on axioms of sets or number. This applies as much to the use of statistical methods as the interpretation of measurement. It may be possible to reduce any problem to first order but any conclusions will then be subject to the assumptions in the reduction. This is particularly insidious in treating open systems as closed. However openness is not just bound up with the concept of order for it contains a deeper logical strand of constructivism as associated with the intuitionism of Brouwer. Boolean logic suffices for a closed system but an open system requires the logic of Heyting (See Figure 7.9 below).

Metaphysical categories have therefore no natural concept of number. Finitary categories as a model relying on the concept of sets has consequently to introduce the concept of number[17]. This is achieved by postulating a Natural Number Object with a recursive definition on arrows comparable to recursive functions generating the set of natural numbers. This requires importing some undefined successor function. While this may be natural in mathematics it is not natural in physics where systems are open either externally or internally. An obvious example is radioactivity where atoms decay according to some preorder and it is not therefore possible to identify a successor before the event of decay. Of course it was to explain such events that the notion of randomness was invented but this is normally dealt with by some theory of statistical probability which leads back to the concept of number and does not provide an exact solution. This lack of a predefined successor is a feature of all open systems and a chief cause of problems of interoperability in global systems.

Open physics lacks a concept of number and therefore questions the use of finitary models in physics. The existence of multiverses must surely be the largest incarnation of the number concept. Listed below are the nine current possible theories identified by Greene [23]. These can also be compared with Barrow's views of multiverses [4].

1. Infinite space may contain a number (possibly an infinity) of universes that may lie beyond our sight.

17 First carried out by Lawvere [34] and now to be found in standard category theory texts, such as [3] at p 177.

2. Uncountable other universes with different characteristics may have been created with ours during a fleeting period of superfast accelerating expansion.
3. String theory suggests our universe is one of many 4-dimensional *brane worlds* floating in a higher-dimensional space-time.
4. A simple cycle of universes with variations in physical laws as possible in string theory.
5. More complex versions of cyclic universes.
6. Quantum mechanics allows/requires many worlds to exist in parallel formed by a branching of the wave function.
7. The universe is a holographic projection.
8. We are just one of a set of artificial universes created in simulation on a super-advanced computer.
9. The philosophical necessity that every possible universe must be realised somewhere.

It is instructive to review Greene's list from the process perspective. The list does not claim to be exhaustive and is an example of undecidability demonstrating how the use of number leads to degeneracy with many possible forms. This degeneracy is well borne out in the thorough examination of n-categories carried out by Leinster [15]. It may well be a comparable defect in string theory that allows variations in physical laws. In process categories, physical laws arise from properties of adjointness whose *bonum esse* is uniqueness. Furthermore about half the items in the list depend on some idea of infinity. But infinity belongs to mathematics, not to physics. It was David Hilbert, the proponent of finitary mathematics, who with the paradox of his Hotel Infinity recognised that infinity is always beyond reach and therefore cannot plausibly exist in physical reality. Infinity in finitary mathematics seems no more than a model of repleteness[18] under

18 Johnstone ([33] at p.3) defines the condition of repleteness as "that any object of the ambient category isomorphic to one in the subcategory is itself in the subcategory".

the free functor in process categories. The last item that postulates every possible universe is also derived from probability theory applied to infinity. That too fails at the Gödel hurdle of 'number'.

As anthropocentric variants on our universe with complicated theories reminiscent of epicycles, multiverses bear an almost Ptolemaic resemblance. The 'super-advanced computer' is a science-fiction vision of current commercial computers. They have not been thought through with respect to quantum computation nor any general attention paid to boundary conditions nor to the relativistic nature of time which Whitehead would carefully respect [61].

8 LOGICAL STRUCTURE OF WORLD REPRESENTATION AS ADJOINTNESS

In terms of natural categories, process is adjointness. This is the formal metaphysics of real existence such that every physical entity in the Universe affects every other. There is at the most a single pair of arrows in opposite directions between any pair of objects. These are limits of all the possible paths around the Universe between any given pair. This limit reduces to a single function as an abstraction in lambda calculus or as a resultant in vector analysis (for first order models lose the resolution of the contravariant pair). There are four levels involving three interfaces. The uppermost level is the intension and the lowest is the extension corresponding respectively to the global and the local. The intermediate interface connects intension and extension, that is snaps the local into the global for all time and space. Any set-theoretic approach finds this latter mechanism, which is essential to all studies of globalisation and interoperability, very difficult if not impossible as recognised by Russell's paradox.

Nevertheless in finitary categories the mathematics of adjointness has been developed in this concept termed a Cartesian closed category, derived as an abstraction of the Cartesian product but this description from historic origins may by its simplicity mislead as to its great power and content. The finitary approach is to distinguish the two properties of Cartesian closed and locally Cartesian closed but in process categories it is that natural distinction between intension and extension. This paper provides an introduction to that formal description of the mathematical structure of the World as found in nature.

To the global/local distinction must be added the stationary against the non-stationary. Both the static and the dynamic are formally representable and accessible in the logic of natural categories. Process relates not just to the non-stationary but subsumes both the static and the dynamic. One is contained in the other but which way round? Such problems, like Zeno's paradox of the arrow's dynamic flight consisting of only static positions are avoided in the 17th century French logic school of the Port Royal [1] (harking back to Aristotle's first and second intentions) by distinguishing the intension from the extension. Aristotle referred to them as first and second intentions. Because of their extended meaning these terms were recognised in the subject of logic by retaining the older spelling with an "s" rather than a "t". When the old subject of logic was superseded around 1900 by symbolic logic based on set theory, the intension/extension relationship became rather lost until the development of computer programming revived it with the need for rigorous typing.

The intension-extension relationship is recursive; thus in the diagram of Figure 7.4 metaphysics is the intension for reality as its extension and reality itself becomes the intension for models as possible extensions. In the natural categories of metaphysics process is adjointness. This is no more than the formal metaphysics of real existence that every physical entity in the Universe affects every other. There is at the most a single pair of arrows in opposite directions between any pair of objects. These are limits of all the possible paths around the Universe between any given pair. This limit is that of the preorder in Figure 7.3. Mathematical categories other than the Cartesian closed are possible but process categories being derived from physics only recognise the existence of Cartesian closed categories which has the property of adjointness. Every object is the domain of a covariant arrow and the co-domain of a contravariant arrow. This recursive structure of intension/extension applies at any level but is best studied between a pair of categories (identity functors $\mathbf{1}_F$ and $\mathbf{1}_G$) where adjointness of the pair of arrows (F and G, contravariant to one another)

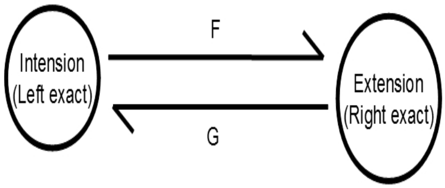

Figure 7.5: Adjointness $F \dashv G$

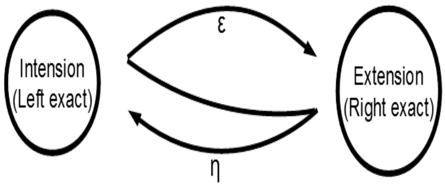

Figure 7.6: Adjointness expressed with natural transformations η and ϵ

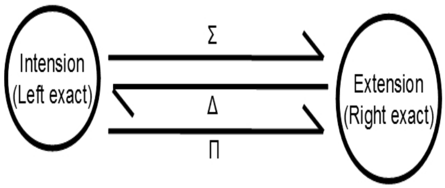

Figure 7.7: Adjointness $\Sigma \dashv \Delta \dashv \Pi$

induce a monad consisting of a triple $< T, \eta, \mu >$ and a co-monad consisting of the co-triple $< S, \epsilon, \delta >$. Figure 7.5 shows the adjointness between the categories, intension and extension.

Each arrow has a dual role. F is the contingent arrow of intension and the determinant arrow of extension while G is the contingent arrow of extension and the determinant arrow of intension. T is just the composition GF and S the composition FG. Each of these compositions may be compared in Figure 7.6 at the next level up with the contribution they make to their respective identity functors by means of the creative unit of adjunction $\eta : \mathbf{1}_F \longrightarrow GF$; and the qualitative co-unit of adjunction $\epsilon : FG \longrightarrow \mathbf{1}_G$. Comparison at the even higher level of order is provided by the unit of potentiality $\mu : T^2 \longrightarrow T$; and its co-unit $\delta : S \longrightarrow S^2$. There are special cases of the latter two which may be interpreted [46] as in the 'dimension of time' with the unit of anticipation where potentiality is by hindsight and the co-unit of anticipation by foresight. Although there are never more than two basic adjoint functors $F \dashv G$, the combined composition of their two compositions T and S may be resolved into the three basic functors of Figure 7.7 to be found in standard category theory texts, where Σ is the existential qualifier, Π the universal quantifier and Δ the stability diagonal pullback functor. The interplay of left and right adjointness with left and right exactness is a little subtle [26] and can be better understood in the exploded diagram of Figure 7.8 which is repeated in Figure 7.9 to show an exploded view of the natural intuitionistic logical structure of the Cartesian closed category.

9 THE NATURAL WORLD STRUCTURE AS A CARTESIAN CLOSED CATEGORY

Relationships in nature are therefore all explicable in process categories with this single concept of adjointness [14] that consists only of a pair of contravariant arrows inducing a monad. In finitary categories the mathematics of adjointness has been developed in what is termed a Cartesian Closed Category, derived as an abstraction of the Cartesian product but this description from historic origins may

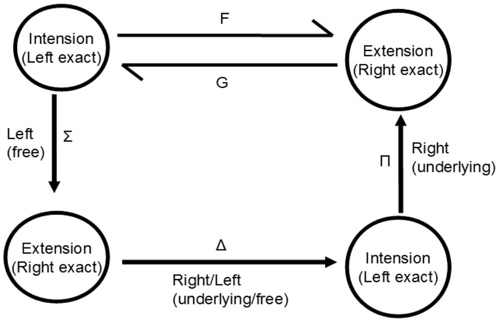

Figure 7.8: Explosion $\Sigma \dashv \Delta \dashv \Pi$ of the Arrow Functors of Adjointness $F \dashv G$

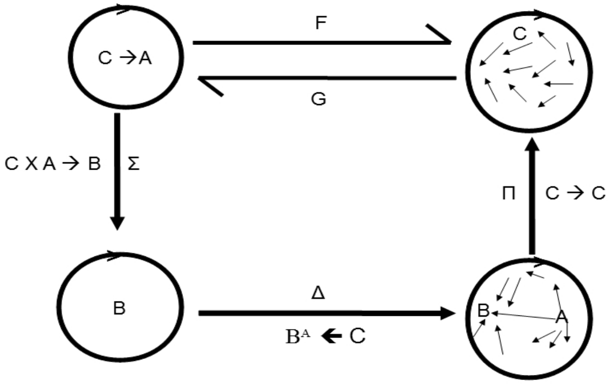

Figure 7.9: Intuitionistic structure of the Cartesian closed category: Exploded view of Heyting logic.

by its simplicity mislead as to its great power and content. The finitary approach is to distinguish the two properties of Cartesian closed and locally Cartesian closed but in process categories it is that natural distinction between intension and extension that provides a formal description of the mathematical structure of the World as found in nature. It is the simple principle that everything in the world is related to everything else in the world that provides the formal structure of the relationship relevant to any scientific study or technological application requiring an understanding of these relationships.

An early example is the representation of information in computers, that needed some implementable model of real-world relationships. Some variation of the hierarchical was possibly the most common structure attempted in different knowledge systems. But the most successful measured by the volume of commercial transactions was by far the simple relational model based on lists or tables manipulated as sets embodying an intension/extension relationship.

The Cartesian closed category (CCC) is a fundamental category of category theory. Its features and their definitions are to be found in its standard textbooks but most if not all come from the stationary viewpoint of set theory, not from process. That set theory itself does not rest on unequivocal foundations may raise few problems in pure mathematics where axioms may be defined at will and may well be adequate too in applied mathematics to a first order. However, many problems requiring mathematical solutions today arise in more complex situations. Transactions in information systems [43] are a case in point as of the nature of process. Thus a common approach in databases [14] is to adopt the principles under the acronym ACID stating the requirements for Atomicity, Consistency, Isolation and Durability. The aim is to ensure that a transaction involving a series of operations is indivisible, enforces all rules, provides results only on termination and guarantees to hold the results under any circumstances. The transaction concept has been implemented efficiently on many database systems but in information systems as a whole the idea lacks the abstraction needed for successful business modelling. The alternative approach in natural philosophy is that of process as explored in the 20th century [28].

While in the formal language of category theory the world may be described as 'Cartesian-closed', this term may give a false impression that it has a Cartesian coordinate system which is unfortunate but the phrase has arisen historically in that context because it embodies the fundamental concept of the Cartesian product. In fact it is much more than a simple product and these terms need to be examined further. For while natural categories and metaphysics provide us with a process structure for the world, we can only begin to investigate it here. Intension and extension alternate in a preorder, that is with an arbitrary beginning of an intension with an extension which itself becomes an intension of the next extension and so on as in Figure 7.10 [29].

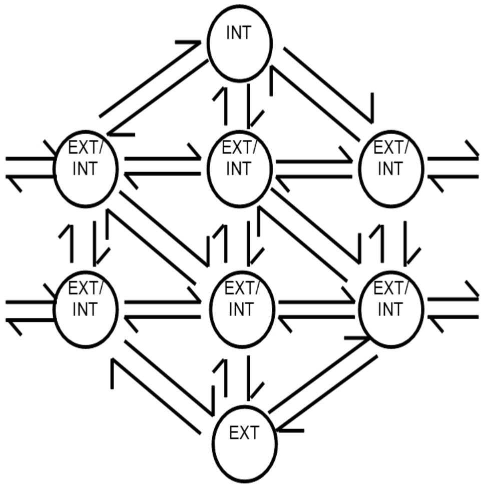

Figure 7.10: Alternate Intension/Extension Pairs in Nature

10 THE TOPOS: ARCHETYPE OF NATURAL WORLD

The archetype of the natural world is the topos, in its early days formally defined as a Cartesian closed category with subobject classifiers and informally as a generalised set. Johnstone in his preface to [33] lists thirteen alternative descriptions that have been applied to the topos (pp.viii *et seq*). Many of them like for instance "A topos is a generalised space" still carry hangovers from sets. We would recommend as an informal definition: "The category of categories of categories". To some this may only confirm categories as "abstract nonsense" but it is accurate and and makes explicit the recursion. The topos sums up all that we have said in this paper. It is the ultimate intension existing as an identity natural transformation in any extension given by the internal categories, subject to the locally Cartesian closed condition with the preorder structure and an intuitionistic logic that is the Heyting and which is more general than the Boolean. There is a unique arrow from the source of the World to every object in it and a unique limiting arrow between any pair of objects.

To satisfy its holistic nature the World must emerge top-down. That is to say no more than that if the Big Bang happened it potentially contained everything that ever existed[19]. However it is easier to explain bottom-up by treating the role of the arrow as a natural expression of process with an identity arrow as intension and a distinguishable valued arrow for extension. Nevertheless while in natural category theory the simplest identity arrow may be treated as an object, it is convenient to begin with a category of three composing objects as a generalisation of any possible category. This is shown in Figure 7.11 with the next higher identity arrow (the functor) composing extensional arrows between objects. The next higher identity arrow is the locally Cartesian closed natural transformation composing categories with ordinary functors as extensional arrows between categories as shown in Figure 7.12. The highest level arrow is also a natural transformation which composes structures of categories and functors. It is this identity natural transformation that constitutes the full Cartesian closed category of a topos as in Figure 7.13. However, the natural arrow is double-headed as a composition of the adjoint functors but with a parity as previously discussed above. Although as just explained it may be easier to understand these diagrams bottom-up in the way that models are usually built-up, nevertheless process can only exist as a whole and the full diagram represents a natural

19 formally (μ, δ) in the description above for the monad/comonad.

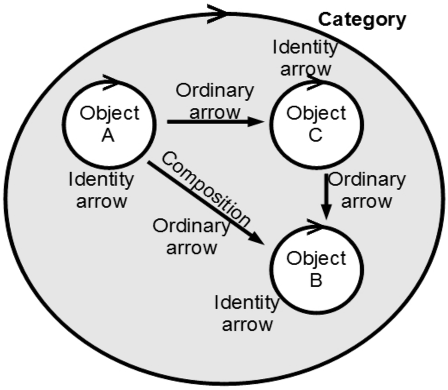

Figure 7.11: A category consists of ordinary arrows composing between identity arrows as objects

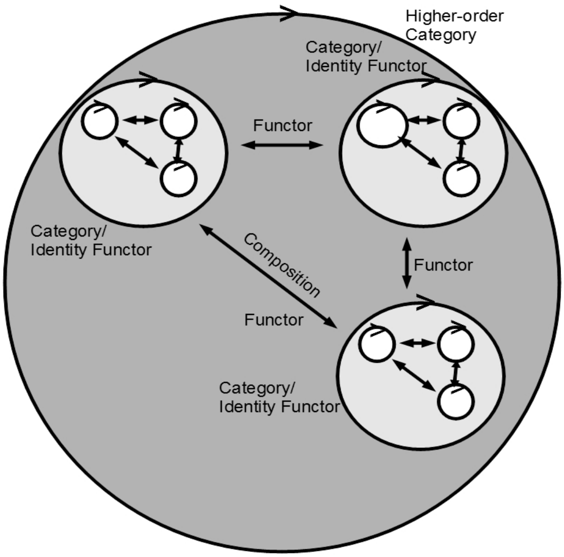

Figure 7.12: Functors between Categories compose to form higher-order categories. A category is just an identity functor

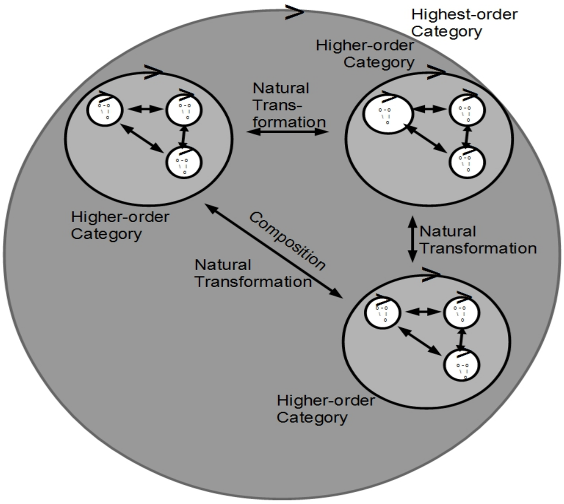

Figure 7.13: Natural Transformations of Composing Functors themselves compose in the highest possible category, a Topos

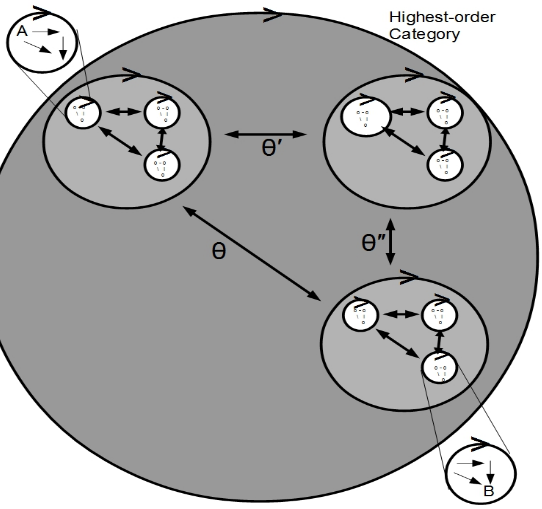

Figure 7.14: A topos showing natural path from any object A to any object B

occasion or "actual event" as first introduced by Whitehead [61] and discussed by McHenry [38].

The whole is just a recursive system with closure at four levels consisting of three open interfaces. Figure 7.13 shows the three interfaces for composing arrows (ordinary, functor, natural transformation) with the four levels (identity arrow, identity functor/category, higher-order identity functor/category, identity natural transformation/topos). The diagram shows well the natural recursive nature of the structure. It also demonstrates connectivity from any object to any other object. It is possible therefore, as shown in Figure 7.14, to get from any object A to any object B directly: $B = \theta A$, or indirectly with possible local variations through any other internal path: $\theta'' \circ \theta' A = B$. This is a natural structure because it is obtained from simple induction applied to the notion of process without any assumptions.

11 CONCLUSIONS

Only a few 'simple' concepts are therefore needed: the World is a topos with monadic objects related by contravariant functors with natural transformations as units of adjunction. These are sufficient to identify formally the Whiteheadian vocabulary of the likes of the ontological principle, actual entities and occasions, eternal objects, congrescence, creative and emotive advance of becoming, public and private, prehensions, nexus, primordial nature, emergence, etc, together with their other postmodern counterparts. The details of these will be pursued at a later date.

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